3/14/07 Lecture 19 outline

• Renormalization of QED. The 1PI photon 1-point amplitude is $i\Pi^{\mu\nu}(p)$. Leading contribution, from electron positron loop is

$$i\Pi_2^{\mu\nu}(p) = (-ie)^2(-1)\int \frac{d^4k}{(2\pi)^4}\mathrm{tr}[\gamma^\mu \frac{i}{\not k - m}\gamma^\nu \frac{i}{\not k + \not p - m}].$$

• Ward identity: $p_\mu \Pi^{\mu\nu} = 0$. Special case of

$$k_\mu \mathcal{M}^\mu(k; p_1, \dots p_n; q_1 \dots q_n) = e\sum_i \mathcal{M}_0(p_1 \dots p_n; q_1 \dots q_i - k, \dots) - \mathcal{M}_0(p_1, \dots p_i + k, \dots; q_1 \dots q_n).$$

The RHS vanishes in S-matrix elements: it doesn't have the right poles to survive LSZ.

Using Ward identity, show $\Pi^{\mu\nu}(p) = (p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi(p^2)$.

• Full propagator:

$$\frac{-ig_{\mu\nu}}{p^2} + \frac{-ig_{\mu\rho}}{p^2}\Pi^{\rho\sigma}(p)\frac{-ig_{\sigma\nu}}{p^2} + \dots$$

Sum the geometric series, get

$$\frac{-i}{p^2(1-\Pi(p^2))}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \frac{-i}{p^2}\left(\frac{p_\mu p_\nu}{p^2}\right).$$

The terms with $p_\mu p_\nu$ will vanish in full amplitudes (via Ward). Note photon stays exactly massless (again, thanks to Ward). But $Z_3 = 1/(1-\Pi(0))$ renormalization of pole at $p^2 = 0$.

• Evaluate $\Pi_2(p^2)$ using dimreg, get

$$\Pi_2(0) = \frac{-2\alpha}{3\pi\epsilon} + \text{finite}.$$

Where $\alpha = e^2/4\pi$ is the fine structure constant.

• 1PI electron propagator $= -i\Sigma(\not p)$. To one loop (virtual photon exchange)

$$-i\Sigma_2 = (-ie)^2\int \frac{d^4k}{(2\pi)^4}\gamma^\mu \frac{i}{\not p - m}\gamma_\mu \frac{-i}{(p-k)^2}.$$

Get full electron propagator via summing geometric series to be

$$\frac{i}{\not p - m - \Sigma}$$

So the location of the pole is shifted if $\Sigma(\not p = m)$ is non-zero, and the residue of the pole is shifted if $\frac{d}{d\not p}\Sigma(\not p = m)$ is non-zero. Define

$$Z_2^{-1} = 1 - \frac{d\Sigma}{d\not p}|_{\not p = m}.$$

So the full propagator near the pole is

$$S(p) = \frac{i}{\not p - m - \Sigma} \to \frac{iZ_2}{\not p - m}.$$

• 1PI vertex for electron interacting with photon, $-ie\Gamma^\mu(p', p)$. The tree-level term is $-ie\gamma^\mu$. The photon has momentum $q = p' - p$. Can show that Lorentz and kinematic structure is such that

$$Z_2\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2),$$

where $\sigma^{\mu\nu} = \frac{1}{2}i[\gamma^\mu, \gamma^\nu]$ and F_i are "form factors." The electron has magnetic moment $\vec{\mu} = g(e\vec{S}/2m)$, with $g = 2 + 2F_2(0)$. The diagram for $F_2(0)$ at one-loop is convergent, and yields $F_2(0) = \alpha/2\pi$. The diagram for $F_1(q^2)$ is divergent at $q^2 = 0$. Define $\Gamma^\mu(q^2 = 0) = Z_1^{-1}\gamma^\mu$.

• The Ward identity gives

$$S(p_k)[-iek_\mu\Gamma^\mu(p_k, p)]S(p) = e(S(p) - S(p_k)).$$

So

$$-ik_\mu\Gamma^\mu(p_k, p) = S^{-1}(p_k) - S^{-1}(p)$$

This gives $Z_1 = Z_2$. Thus $F_1(0) = 1$.

• Bare and renormalized fields, and counterterms. $\psi_B = Z_2^{1/2}\psi_R$, $A_B^\mu = Z_3^{1/2}A_R^\mu$, $e_B Z_2 Z_3^{1/2} = e_R Z_1$. $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{c.t.}$.

$$\mathcal{L}_R = -\frac{1}{4}F_{R\mu\nu}F_R^{\mu\nu} + \bar{\psi}_R(i\not\partial - e_R \not A_R - m_R)\psi_R,$$

$$\mathcal{L}_{ct} = -\frac{1}{4}\delta_3(F_{R\mu\nu})^2 + \bar{\psi}_R(i\delta_2\not\partial - \delta_1 e_R \not A_R - \delta_m)\psi_R.$$

Where $\delta_1 = Z_1 - 1$, $\delta_2 = Z_2 - 1$, $\delta_3 = Z_3 - 1$, and $\delta_m = Z_2 m_0 - m$. We have

$$e_B Z_2 Z_3^{1/2} = e_R Z_1,$$

where the Z_1 will cancel the Z_1^{-1} in $\Gamma^\mu(q^2 = 0) = Z_1^{-1}\gamma^\mu$.

• Gauge invariance requires $Z_1 = Z_2$, since then $\delta_1 = \delta_2$ and the counterterm pieces have the same gauge invariance. Sure enough, direct calculation shows $Z_1 = Z_2$ (to all orders in perturbation, theory, and exactly)! So $e_R = \sqrt{Z_3}e_0 = e_{phys}$. Shows that renormalized charge is same for all species (e.g. electron and muon and anti-proton all have exactly the same effective charge).