2/21/07 Lecture 12 outline

• The quantity appearing in our functional integral is S_B. We split $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{c.t.}$. For fixed physics, the LHS is some fixed quantity. How we split it up on the RHS depends on our renormalization scheme.

• Last time: $\lambda\phi^4$. Recall

$$\mathcal{L}_{c.t.} = \tfrac{1}{2}(Z_\phi - 1)\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}(m_B^2 Z_\phi - m^2)\phi^2 - (\lambda_B Z_\phi^2 - \lambda\mu^\epsilon)\frac{1}{4!}\phi^4.$$

Define $\delta_Z \equiv Z_\phi - 1$, $\delta_m = m_B^2 Z_\phi - m^2$, $\delta_\lambda \mu^\epsilon = \lambda_B Z_\phi^2 - \lambda\mu^\epsilon$. Recall also that

• Last time: fixed renormalization scheme by

$$\Pi'(m^2) = 0, \qquad \frac{d\Pi'}{dp^2}|_{p^2=m^2} = 0, \qquad \tilde{\Gamma}^{(4)}|_{s=\mu} = -\lambda$$

Last time we took $\mu = 4m^2$. We could also change the renormalization point μ.

• There are two other renormalization schemes worth mentioning. Minimal subtraction (MS) where we choose the counterterms to remove the $1/\epsilon$ poles, and nothing else. A variant is $\overline{MS}$, where one replaces

$$\frac{\Gamma(2-\frac{1}{2}D)}{(4\pi)^{D/2}(m^2)^{2-\frac{1}{2}D}} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \log(4\pi/m^2)\right)$$

with

$$\frac{1}{16\pi^2}\log(M^2/m^2),$$

for some arbitrary mass parameter M. The apparent freedom to define things many different ways always cancels out at the end of the day, when one relates to physical observables. Different choices have different benefits along the way.

• Let's consider $\lambda\phi^4$ in MS. Recall that we have

To one loop, we have

$$\delta_m = \frac{\lambda m^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_\lambda = \frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_Z = 0.$$

Now consider the propagator to two loops. Diagram 1 is a one-loop diagram with the 1-loop $\delta\lambda$ counterterm at the vertex. Diagram 2 is a one-loop diagram with the 1-loop δ_m counterterm on the internal propagator. Diagram 3 is a two-loop diagram which looks like a double-scoop of the 1-loop diagrams. Diagram 4 is the one from your HW. Diagram 5 have no loops, but an insertion of the 2-loop δ_m and δ_Z counter terms. Let's consider the pole terms in the diagrams. Diagram 1 gives

$$i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{3}{2}\left(\frac{2}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{m^2}{4\pi\mu^2}+\frac{1}{\epsilon}-\frac{\gamma}{\epsilon}\right)+\mathcal{O}(\epsilon^0)$$

Diagram 2 gives

$$i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{1}{2}\left(\frac{2}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{m^2}{4\pi\mu^2}-\frac{\gamma}{\epsilon}\right)+\mathcal{O}(\epsilon^0)$$

Diagram 3 gives

$$-i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{1}{2}\left(\frac{2}{\epsilon^2}-\frac{2}{\epsilon}\ln\frac{m^2}{4\pi\mu^2}+\frac{1}{\epsilon}-\frac{2\gamma}{\epsilon}\right)+\mathcal{O}(\epsilon^0)$$

Diagram 4 gives

$$i\frac{\lambda^2}{(16\pi^2)^2}\left(-\frac{m^2}{\epsilon^2}+\frac{1}{\epsilon}\left(m^2\ln\frac{m^2}{4\pi\mu^2}+\frac{1}{12}p^2+(\gamma-\frac{3}{2}m^2)\right)\right)$$

Diagram 5 are the two-loop counterterms, $i\delta_Z^{(2)}p^2-i\delta_m^{(2)}$. We should then take for the 2-loop contributions to the counterterms

$$\delta m^{(2)}=\frac{\lambda^2}{(16\pi^2)^2}\left(\frac{2}{\epsilon^2}-\frac{1}{2\epsilon}\right)m^2,$$

$$\delta_Z^{(2)}=-\frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{\epsilon}.$$

The terms involving $\ln m^2/4\pi\mu^2$ all cancel. This happens for all loops. MS is a mass independent scheme, in that $\delta\lambda$, δZ, and $\delta m/m^2$ are independent of m and μ.

• Renormalizability: all divergences cancelled by counter terms of the same form as original $\mathcal{L}$. This would not be the case for e.g. $\lambda\phi^6$. Even for $\lambda\phi^4$, it is quite non-trivial. For example, in doing 2 loops, there could have been some term from one loop diagrams, with counter terms, leading to $\frac{1}{\epsilon}\ln p^2$, which could not be cancelled by a counterterm in our lagrangian. Sometimes individual diagrams indeed behave like that. But the coefficients of all such terms sum to zero.

• Renormalized and bare Greens functions.

$$\tilde{\Gamma}_B^{(n)}(p_1,\ldots p_n;\lambda_B,m_B,\epsilon)=Z_\phi^{-n/2}\tilde{\Gamma}_R^{(n)}(p_1,\ldots p_n;\lambda_R,m_R,\mu,\epsilon).$$

For fixed physics, the LHS is some fixed quantity. The RHS depends on the renormalization point μ and the scheme. The LHS does not! This leads to what is known as the renormalization group equations, which state how the renormalized quantities must vary with μ. Rewrite above as

$$Z_\phi^{n/2}\tilde{\Gamma}_B^{(n)}(p_1,\ldots p_n;\lambda_B,m_B,\epsilon)=\tilde{\Gamma}_R^{(n)}(p_1,\ldots p_n;\lambda_R,m_R,\mu,\epsilon).$$

Now the RHS is finite, so the LHS must be too. So we can take $\epsilon\to 0$ without a problem.

End here. Next time:

Take $d/d\ln\mu$ of both sides, and use $d\Gamma_B/d\mu=0$. This gives

$$\left(\frac{\partial}{\partial\ln\mu}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}+\gamma_m m_R\frac{\partial}{\partial\ln m_R}-n\gamma\right)\tilde{\Gamma}_R^{(n)}(p_1,\ldots p_n;\lambda_R,m_R,\mu)=0$$

Here

$$\beta(\lambda)\equiv\frac{d}{d\ln\mu}\lambda_R$$

$$\gamma=\tfrac{1}{2}\frac{d}{d\ln\mu}\ln Z_\phi$$

$$\gamma_m=\frac{d\ln m_R}{d\ln\mu}.$$

This is the Callan-Symanzik equation.

• Understand what β and γ mean: the bare quantities are some function of the renormalized ones and epsilon. E.g. for $\lambda\phi^4$ in MS we have

$$\lambda_B=\mu^\epsilon(\lambda+\delta_\lambda)=\mu^\epsilon(\lambda+\sum_k a_k(\lambda)\epsilon^{-k}),$$

where we found $a_1(\lambda)=+3\lambda^2/16\pi^2$ to one loop. The bare parameter λ_B is independent of μ, whereas λ depends on μ, such that the above relation holds. Take $d/d\ln\mu$ of both sides,

$$0=\epsilon(\lambda+\sum_k a_k\epsilon^{-k})+\mu\frac{d\lambda}{d\mu}(1+\sum_k a_k'(\lambda)\epsilon^{-k}).$$

It follows that

$$\beta(\lambda)=\lim_{\epsilon\to 0}\frac{d\lambda}{d\ln\mu}=-(1-\lambda\frac{d}{d\lambda})a_1(\lambda).$$

We find for $\lambda\phi^4$

$$\beta(\lambda)=\frac{3\lambda^2}{16\pi^2}+\mathcal{O}(\lambda^3).$$

Integrating, this gives

$$\lambda=\lambda_0\left(1-\frac{3}{16\pi^3}\lambda_0\ln(\mu/\mu_0)\right)^{-1}.$$