Week 8 Homework (Due Nov. 16)

1. Use the following conventions: $\eta_{\mu\nu}$ =diag$(-1,1,1,1)$, $\epsilon^{0123} = 1$. Define the Hodge dual of a two index antisymmetric tensor $F_{\mu\nu}$ by $(*F)_{\mu\nu} = c\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, with c a real normalization constant such that $** = -1$.

a. Find the constant c.

b. Let $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the electromagnetic field strength. Define a "dual" electromagnetic field strength by $\widetilde{F}_{\mu\nu} \equiv (*F)_{\mu\nu}$. Express the dual electric and magnetic fields, $\widetilde{\mathbf{E}}$, and $\widetilde{\mathbf{B}}$ in terms of the original fields $\mathbf{E}$ and $\mathbf{B}$. Using this, rewrite Maxwell's equations in terms of $\widetilde{\mathbf{E}}$, and $\widetilde{\mathbf{B}}$.

c. What linear combinations of $\mathbf{E}$ and $\mathbf{B}$ are eigenstates of the $*$ operation, and with what eigenvalues?

2. As in the lectures, we take $\sigma^\mu_{\alpha\dot{\alpha}}$, with $(\sigma^0)_{\alpha\dot{\alpha}} = -\delta_{\alpha\dot{\alpha}}$ and the other $(\sigma^i)_{\alpha\dot{\alpha}}$ the usual Pauli matrices. For any vector v_μ, we define $v_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} v_\mu$.

a. Define $\overline{\sigma}^{\mu\dot{\alpha}\alpha} \equiv \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}}$. Verify that $v^\mu = -\frac{1}{2}\overline{\sigma}^{\mu\dot{\alpha}\alpha}v_{\alpha\dot{\alpha}}$.

b. Consider a general two index antisymmetric tensor $F_{\mu\nu}$. Convert the vector indices to spinor indices and write out the resulting $F_{\alpha\beta}$ and $F_{\dot{\alpha}\dot{\beta}}$ as matrices, with components given in terms of the original $F_{\mu\nu}$.

c. In particular, when $F_{\mu\nu}$ is the electromagnetic field strength, write the $F_{\alpha\beta}$ and $F_{\dot{\alpha}\dot{\beta}}$ in terms of the components of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. Comparing with part (1c), note that $F_{\alpha\beta}$ and $F_{\dot{\alpha}\dot{\beta}}$ are eigenstates of $*$, and find the corresponding eigenvalues.

3. Consider a chiral superfield Φ and anti-chiral superfield $\overline{\Phi} = \Phi^\dagger$, and the superspace differential equation

$$\overline{D}_{\dot{\alpha}}\overline{D}^{\dot{\alpha}}\overline{\Phi} - m\Phi = 0.$$

a. The second term is chiral; verify that the first term is also chiral (annihilated by $\overline{D}_{\dot{\beta}}$).

b. Expand out both sides in θ^α and $\overline{\theta}^{\dot{\alpha}}$. Set each component to zero and verify that you get the equations of motion for a massive boson and fermion.