Week 6 Homework (Due Nov. 2)

Consider 2+1 dimensions, with coordinates x^μ, $\mu = 0, 1, 2$. Let the Lorentz boost and rotation matrix act on a vector via $x^\mu \to R^\mu_\nu x^\nu$ with $R = \exp\left(i(\phi_0 T^0 + \phi_1 T^1 + \phi_2 T^2)\right)$, and

$$T^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad T^1 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad T^2 = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

1. Write out the Lorentz matrices $R_0 = \exp(i\phi_0 T^0)$, $R_1 = \exp(i\phi_1 T^1)$, and $R_2 = \exp(i\phi_2 T^2)$ and identify them as rotations or boosts.

   Now consider the 2-component spinor representation, where $\psi^\alpha \to R^\alpha_\beta \psi^\beta$, with $R = \exp(i\phi_\mu T^\mu)$ and $(T^\mu)^\beta_\alpha = \frac{i}{2}(\Gamma^\mu)^\beta_\alpha$ and $\Gamma^0 \equiv i\sigma_2$, $\Gamma^1 \equiv \sigma_1$, $\Gamma^2 \equiv \sigma_3$, with $(\sigma_i)^\beta_\alpha$ the usual Pauli matrices. We have defined the Γ^μ to be purely real.

2. Using the fact that $[\frac{1}{2}\sigma_i, \frac{1}{2}\sigma_j] = i\epsilon_{ijk}(\frac{1}{2}\sigma_k)$, find the structure constants $f_{\mu\nu\gamma}$ in $[T^\mu, T^\nu] = if_{\mu\nu\gamma}T^\gamma$ using the above spinor representation. Compute $[T_0, T_1]$ again using the above vector representation and verify that it gives the same structure constants as found for the spinor representation.

3. Write out the Lorentz rotation matrices $R_0 = \exp(i\phi_0 T^0)$, $R_1 = \exp(i\phi_1 T^1)$, and $R_2 = \exp(i\phi_2 T^2)$ in this spinor representation. Verify that each of these three basic rotations/boosts is an element of the group $SL(2, R)$.

4. Take $\epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21} = +1$. With this choice of signs $\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = \delta^\alpha{}_\beta$. Let $\Gamma^\mu_{\alpha\beta} \equiv (\Gamma^\mu)^\gamma_\alpha \epsilon_{\gamma\beta}$, with $(\Gamma^\mu)^\beta_\alpha$ as defined above in terms of the usual Pauli matrices. Write out, as a vector, the $\mu = 0, 1, 2$ components of $V^\mu \equiv \Gamma^\mu_{\alpha\beta}\psi^\alpha\chi^\beta$ for two spinors ψ^α and χ^β. Rotate $\psi^\alpha \to R^\alpha_\gamma \psi^\gamma$ and $\chi^\beta \to R^\beta_\gamma \chi^\gamma$ for $R = R_1 = \exp(i\theta T^1)$ as found in problem 3. Plug this into V^μ to find $V^\mu \to R^\mu_\nu V^\nu$ for some R^μ_ν which (hopefully) agrees with the rotation matrix R_1 for vectors, as found in problem 1.

5. The minimal 2+1 dimensional supersymmetry algebra is $\{Q_\alpha, Q_\beta\} = 2\Gamma^\mu_{\alpha\beta}P_\mu$, with the Q_α real. We can represent this as a superspace with two real anticommuting spinor coordinates θ^α via $P_\mu = i\partial_\mu$, $Q_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\Gamma^\mu_{\alpha\beta}\theta^\beta\partial_\mu$, and also $D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\Gamma^\mu_{\alpha\beta}\theta^\beta\partial_\mu$. We take a superfield to be $\Phi = \phi + i\theta^\alpha\psi_\alpha + \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta F$. Consider the action

$$S = \int d^3x d\theta^1 d\theta^2 (-\tfrac{1}{2}\epsilon^{\alpha\beta}D_\alpha\Phi D_\beta\Phi + W(\Phi))$$

   Do the θ integrals and eliminate F to find the Lagrangian.