$4d$ $(\mathcal{N}=1)$ SUSY Alg: $\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu = 2P_{\alpha\dot\alpha}$

$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$

Massive reps: boost to rest frame $P=(-m,0,0,0)$

$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2m\,\delta_{\alpha\dot\alpha}, \quad \{Q_\alpha, Q_\beta\} = 0$

$$\left.\begin{array}{l} |\Omega\rangle \quad \text{with} \quad Q_\alpha|\Omega\rangle = 0 \\ \\ \bar{Q}_{\dot\alpha}|\Omega\rangle \\ \\ \varepsilon^{\dot\alpha\dot\beta}\bar{Q}_{\dot\alpha}\bar{Q}_{\dot\beta}|\Omega\rangle \end{array}\right\} \underline{\text{rep}}$$

e.g. $|\Omega\rangle$ spin $j=0 \longrightarrow 1$ boson d.f.

$\bar{Q}_{\dot\alpha}|\Omega\rangle$ spin $j=1/2 \longrightarrow 2$ fermi d.f.

$\varepsilon^{\dot\alpha\dot\beta}\bar{Q}_{\dot\alpha}\bar{Q}_{\dot\beta}|\Omega\rangle$ spin $j=0 \longrightarrow 1$ bose d.f.

As expected, equal # of bose & fermi d.f.

more generally $|\Omega\rangle$ spin $j \longrightarrow 2j+1$

$\bar{Q}_{\dot\alpha}|\Omega\rangle \quad j+1/2 \oplus j-1/2 \longrightarrow (2j+2)+(2j)$

$\varepsilon^{\dot\alpha\dot\beta}\bar{Q}_{\dot\alpha}\bar{Q}_{\dot\beta}|\Omega\rangle \quad j \longrightarrow (2j+1)$

Massless rep: $P_\mu = (-E, E, 0, 0)$

$\{Q_1, \bar{Q}_{\dot{1}}\} = 4E$ all others $= 0$

$|\Omega\rangle \leftarrow$ annihilated by Q_α

$\bar{Q}_{\dot{1}} |\Omega\rangle$

$|\Omega\rangle \rightarrow$ helicity λ integer or ½ integer

$\bar{Q}_{\dot{1}} |\Omega\rangle$ helicity $\lambda + \frac{1}{2}$

massless multiplet helicities $(\lambda, \lambda + \frac{1}{2})$

CTP conjugate $(-\lambda, -\lambda - 1/2)$

E.g. "chiral multiplet" $\lambda = (0, 1/2) + (0, -1/2)$

$2 \times \lambda = 0 \rightarrow$ complex scalar

$\lambda = \pm 1/2 \rightarrow$ Majorana fermion ψ_α w/ $\psi_\alpha^* = \bar{\psi}_{\dot{\alpha}}$

"Vector multiplet" $\lambda = \pm 1/2, \pm 1$

$\pm 1/2$: Massless Majorana fermion

± 1 : Massless vector boson (e.g. photon)

Spinor conventions (Wess & Bagger, appendix A)

$\varepsilon^{12} = \varepsilon_{21} = 1$. Summation convention: $\begin{cases} \psi\chi \equiv \psi^\alpha \chi_\alpha \\ \bar\psi\bar\chi \equiv \bar\psi_{\dot\alpha} \bar\chi^{\dot\alpha} \end{cases}$

Can show $\psi\chi = \chi\psi$, $\bar\psi\bar\chi = \bar\chi\bar\psi$

$(\chi\psi)^+ = \bar\psi\bar\chi$.

Superspace: $(x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha})$

Supertranslation generator: $G(x, \theta, \bar\theta) = e^{i(xP + \theta Q + \bar\theta\bar Q)}$

$\Rightarrow G(0, \xi, \bar\xi)\, G(x, \theta, \bar\theta) = G(x^\mu + i\theta\sigma^\mu\bar\xi - i\xi\sigma^\mu\bar\theta,$
$\theta + \xi, \bar\theta + \bar\xi)$

Generate
$$x^\mu \to x^\mu + i\theta\sigma^\mu\bar\xi - i\xi\sigma^\mu\bar\theta$$
$$\theta \to \theta + \xi$$
$$\bar\theta \to \bar\theta + \bar\xi$$

Via
$$\xi Q + \bar\xi\bar Q = \xi^\alpha\left(\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\right)$$
$$+ \bar\xi_{\dot\alpha}\left(\frac{\partial}{\partial\bar\theta_{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\beta}\varepsilon^{\dot\beta\dot\alpha}\partial_\mu\right)$$

$$\therefore \quad Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\, \sigma^\mu_{\alpha\dot\alpha}\, \bar\theta^{\dot\alpha}\, \partial_\mu$$

$$\bar Q_{\dot\alpha} = -\frac{\partial}{\partial \bar\theta^{\dot\alpha}} + i\, \theta^\alpha\, \sigma^\mu_{\alpha\dot\alpha}\, \partial_\mu$$

(some sign changes from lowering $\dot\alpha$ index)

Satisfy $\{Q_\alpha, \bar Q_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \equiv 2P_{\alpha\dot\alpha}$

with $P_\mu \to i\partial_\mu$

Also covt. derivaties: $\left\{ \begin{array}{l} D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \\ \bar D_{\dot\alpha} = -\bar\partial_{\dot\alpha} - i\theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\partial_\mu \end{array} \right.$

$\left(\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha},\ \bar\partial_{\dot\alpha} \equiv \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\right)$

$\{D_\alpha, Q_\beta\} = 0 \qquad \{\bar D_{\dot\alpha}, Q_\beta\} = 0$

$\{D_\alpha, \bar D_{\dot\alpha}\} = -2P_{\alpha\dot\alpha}$

Construct superfields $\Phi(x, \theta, \bar\theta)$

need to impose constent to get irrep.

One possibility: reality constant $\to$ vector multiplet later

Another: $\bar D_{\dot\alpha}\Phi = 0 \to$ chiral superfield.

Let $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar\theta$, $\quad \bar D_{\dot\alpha}(y^\mu) = 0$

Chiral superfield: $\Phi = \Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y)$
$+ \theta^2 F(y) = \phi(x) + \sqrt{2}\,\theta\psi(x) + \theta^2 F(x)$
$+ i\theta\sigma^\mu\bar\theta\,\partial_\mu\phi - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\,\sigma^\mu\bar\theta$
$+ \frac{1}{4}\theta^2\bar\theta^2\partial^2\phi$. $\quad$ (Use e.g. $\theta^\alpha\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}\theta^2$)

Likewise, anti-chiral superfield $\bar\Phi$, $D_\alpha\bar\Phi = 0$

$\bar\Phi = \bar\Phi(\bar y, \bar\theta)$, $\quad \bar y \equiv x^\mu - i\theta\sigma^\mu\bar\theta$.

$\Phi^\dagger$ transforms as $\bar\Phi$.

Susy $\mathcal{L}$: $\quad \mathcal{L} = \int d^4\theta\, K(\Phi, \bar\Phi) + \int d^2\theta\, W(\Phi)$
$+ \int d^2\bar\theta\, \bar W(\bar\Phi)$ $\quad$ with $\bar\Phi \equiv \Phi^\dagger$.
$\bar W(\bar\Phi) \equiv W(\Phi)^\dagger$

note $K \to K + f(\Phi) + \bar f(\bar\Phi)$ leaves $\mathcal{L}$ unchanged

<u>Example</u>: $K = \Phi\bar\Phi$, $\quad W = 0$

$\mathcal{L} = \bar F F - \partial_\mu\phi^\dagger\partial_\mu\phi + \bar\psi_{\dot\alpha}\, i\partial^{\dot\alpha\alpha}\psi_\alpha$

EOM $\Rightarrow$ $F = \bar F = 0$

Free complex boson + free 2 compt fermion,
(2 bose + 2 fermi d.f.).

ϕ EOM : $\partial_\mu \partial^\mu \phi = 0$, ψ EOM : $i \partial^{\dot{\alpha}\alpha} \psi_\alpha = 0$

+ h.c. EOM for ϕ^* & ψ^+.

Can write EOM in superspace : $D_\alpha D^\alpha \Phi = 0$

some global symmetries $\Phi \to e^{i\alpha} \Phi \qquad \theta \to \theta$

$\Phi^+ \to e^{-i\alpha} \Phi^+ \qquad \bar{\theta} \to \bar{\theta}$

i.e.

field :	ϕ	ψ_α	F	ϕ^*	$\psi^+_{\dot{\alpha}}$	F^*
charge:	1	1	1	−1	−1	−1

Symmetry commutes with supercharge.
" Non - R symmetry".

Example of R symmetry $\Phi \to \Phi \qquad \theta_\alpha \to e^{-i\phi} \theta_\alpha$

field :	ϕ	ψ_α	F	ϕ^*	$\psi^+_{\dot{\alpha}}$	F^*
Charge :	0	−1	−2	0	1	2

components of chiral superfield have different charges, Supercharge carries charge under Symmetry $\psi_\alpha \sim Q_\alpha(\phi)$ so Q_α has charge −1

$U(1)_R$: Q_α charge -1, $Q^\dagger_{\dot\alpha}$ charge $+1$

$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2P_{\alpha\dot\alpha}$ ← $P_{\alpha\dot\alpha}$ neutral

<u>non</u> R symmetry: Q_α & $\overline{Q}_{\dot\alpha}$ neutral.

More general $\mathcal{L} = \int d^4\theta \, K(\Phi, \overline{\Phi})$ respects

$U(1)_R$: $\Phi \to \Phi$ $\quad \theta \to e^{i\gamma}\theta$ $\quad \bar\theta \to e^{-i\gamma}\bar\theta$

$K \to K$, $d^4\theta \to d^4\theta$ so symm of $\mathcal{L}$.

Also respects non-R symm $U(1)_\Phi$: $\Phi \to e^{i\gamma}\Phi$
if $K(\Phi, \overline{\Phi}) = K(\Phi\overline{\Phi})$.

$K = \overline{\Phi}\Phi \to \mathcal{L} = -\partial_\mu \phi^\dagger \partial^\mu \phi + \psi^\dagger_{\dot\alpha} \, i\partial^{\dot\alpha\alpha} \psi_\alpha$

has vacua with $\langle\phi\rangle$ = arbitrary. These all have unbroken susy. Unbroken susy requires $\langle Q_\alpha(\text{anything})\rangle = \langle \overline{Q}_{\dot\alpha}(\text{anything})\rangle = 0$

but $\phi \neq Q_\alpha(\text{anything})$
$\phi \neq \overline{Q}_{\dot\alpha}(\text{anything})$ so OK to have $\langle\phi\rangle \neq 0$.

General fact: scalar compt of chiral superfield vev doesn't break susy. ψ_α or F vevs would break susy.

So $\mathcal{L} = \int d^4\theta \, \bar{\Phi} \Phi$ has a "moduli space of susy vacua" for any $\langle \phi \rangle \in \mathbb{C}$

complex plane

$\mathrm{Tr}(-1)^F = \infty$ (ill defined)

Consider more general $\mathcal{L} = \int d^4\theta \, K(\Phi_i, \bar{\Phi}_{\bar{j}})$

$i = 1 \cdots n$

Φ_i : 4d Minkowski space $\longrightarrow$ n complex dim'l manifold Σ with coords Φ_i

$$\mathcal{L} = g_{i\bar{j}} \left(-\partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{j}} + \bar{\psi}^{\bar{j}} (-i D) \psi^i \right)$$

$$+ \tfrac{1}{4} R_{i\bar{j}k\bar{l}} \, \psi^i \psi^k \bar{\psi}^{\bar{i}} \bar{\psi}^{\bar{j}}$$

metric of Σ is "Kahler" $g_{ii} = g_{\bar{j}\bar{j}} = 0$

$$g_{i\bar{j}} \equiv \partial^2 K / \partial\phi_i \, \bar{\partial} \bar{\phi}_{\bar{j}}$$

$K =$ "Kahler potential".

$\Gamma^i_{jk} = g^{i\bar{i}} \, g_{j\bar{i},k}$, $\quad R_{i\bar{j}k\bar{l}} = g_{i\bar{j},k\bar{l}} - g_{m\bar{m}} \Gamma^m_{ik} \Gamma^{\bar{m}}_{\bar{j}\bar{l}}$

$$D^{\alpha\dot{\alpha}} \psi_i = \partial^{\alpha\dot{\alpha}} \psi_i + \Gamma^i_{jk} \, \partial^{\alpha\dot{\alpha}} \phi^k \psi^j$$