Can instead impose a Majorana condition that

$\Gamma^\mu \to \pm(\Gamma^\mu)^*$ by a similarity transform

also reduces dimension of rep by factor of 2.

Possible only for $d = 0,1,2,3,4 \mod 8$.

Can impose both Majorana & Weyl only in $d = 2 \mod 8$.

In particular, for $d=4$ Minkowski space $(1,3)$

Majorana spinors $\to 2^{\frac{d}{2}} = 4$ real dim'l,

Write as 2, 2 compt. Weyl spinors

$\psi_\alpha, \bar{\psi}_{\dot\alpha}$ with $(\psi_\alpha)^* = \bar{\psi}_{\dot\alpha}$

$\alpha = 1,2$ $\dot\alpha = 1,2$ The α & $\dot\alpha$ indices

are in the fundamental of $SU(2)_L$ & $SU(2)_R$

in Euclidean version $Spin(4) \simeq SU(2)_L \times SU(2)_R$.

In Minkowski space $Spin(1,3) \simeq SL(2,\mathbb{C})$

With Lorentz gp = 2×2 complex matrices w/

$\det = 1 \to 3$ complex $= 6$ real dim'l

$= 3$ rotation angles + 3 boosts.

$\psi_\alpha \to M_\alpha^{\ \beta}\, \psi_\beta$ $\quad \bar{\psi}_{\dot\alpha} \to (M^*)_{\dot\alpha}^{\ \dot\beta}\, \bar{\psi}_{\dot\beta}$

ψ_α left handed spinor $(1/2, 0)$

$\bar{\psi}_{\dot{\alpha}}$ right handed spinor $(0, 1/2)$

a^μ 4 vector $\longrightarrow$ $(1/2, 1/2)$

e.g. $a_{\alpha\dot{\alpha}} = a_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \begin{pmatrix} (-a_0 + a_3) & (a_1 - i a_2) \\ (a_1 + i a_2) & (-a_0 - a_3) \end{pmatrix}$

Can write Dirac spinor as

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

(Where $\bar{\sigma}^{\mu\,\dot{\alpha}\alpha} \equiv \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} \sigma^\mu_{\beta\dot{\beta}}$)

A Majorana spinor is a Dirac spinor with

$\bar{\chi}^{\dot{\alpha}} \equiv \bar{\psi}^{\dot{\alpha}} \equiv (\psi^\alpha)^*$.

Dirac eqn for Weyl spinor: $\partial^{\alpha\dot{\alpha}} \psi_\alpha = 0$

follows from $\mathcal{L} = i \bar{\psi}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \psi^\alpha$

$\partial_{\alpha\dot{\alpha}} \equiv \partial_\mu \sigma^\mu_{\alpha\dot{\alpha}}$

$$\mathcal{L} = i\,\psi^{\dagger}_{\dot\alpha}\,\partial^{\dot\alpha\alpha}\,\psi_\alpha$$

CTP conjugate of ψ_α, all quantum # charges reversed, so $\mathcal{L}$ is indeed invt. If the theory is "vector like" it respects C & P separately which implies for every ψ_α, there is some $\tilde\psi_\alpha$, which is also left handed, but has opposite charges. E.g. electron & Positron respect C & P separately:

$$\text{electron:} \quad \psi_\alpha \xrightarrow{CPT} \psi^{\dagger}_{\dot\alpha} \quad \text{(positron)}$$

$$C \quad \text{positron:} \quad \tilde\psi_\alpha \longrightarrow \tilde\psi^{\dagger}_{\dot\alpha} \quad \text{(electron)}$$

The neutrino does not

$$\text{neutrino} \quad \nu_\alpha \xrightarrow{CPT} \nu^{\dagger}_{\dot\alpha} \quad \text{(antineutrino)}$$

no right handed neutrino or left handed anti neutrino: $\nexists\ \tilde\nu_\alpha$ related to ν_α by C.

A theory, such as neutrino, which is not "vector like" is "chiral".

In a vectorlike theory can add mass terms

$$\mathcal{L} = \psi^{\dagger}_{\dot{\alpha}} (i \partial^{\dot{\alpha}\alpha}) \psi_\alpha + \tilde{\psi}^{\dagger}_{\dot{\alpha}} (i \partial^{\dot{\alpha}\alpha}) \tilde{\psi}_\alpha$$

$$- m \varepsilon^{\alpha\beta} \psi_\alpha \tilde{\psi}_\beta - m^* \varepsilon^{\dot{\alpha}\dot{\beta}} \tilde{\psi}^{\dagger}_{\dot{\alpha}} \psi^{\dagger}_{\dot{\beta}}$$

In a chiral theory, mass terms generally not allowed on symmetry grounds:

$\Delta\mathcal{L} = -m \psi_\alpha \psi_\beta \varepsilon^{\alpha\beta}$ respects Lorentz but generally violates some charge conservation. E.g. QED with no left handed positron $\tilde{\psi}_\alpha$, $\Delta\mathcal{L} \sim \psi_\alpha \psi_\beta \varepsilon^{\alpha\beta}$ wouldn't conserve electric charge. Couldn't add mass term.

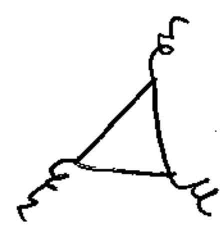
Even when mass terms are allowed by gauge charge conservation, they generally violate some global symmetries. E.g. mass terms in above $\mathcal{L}$ violate global chiral symmetry which rotates ψ but not $\tilde{\psi}$. Such chiral global symmetries, when they're $U(1)$ symmetries, can be anomalous - more later on anomalies.

Standard model each generation

	$SU(3)$	$SU(2)$	$U(1)_Y$
Q_α	3	2	$1/3$
$\tilde{u}_\alpha$	$\bar{3}$	1	$-4/3$
$\tilde{d}_\alpha$	$\bar{3}$	1	$2/3$
L_α	1	2	-1
$\tilde{e}_\alpha$	1	1	2

- $SU(3)$ has all vectorlike reps

$SU(2)$ & $U(1)_Y$ reps are chiral.

Anomaly constraint e.g. 

$\sim\sim$ = $U(1)_Y$ gauge field

— = fermions charged under $U(1)_Y$

- $$\text{anomaly} \sim \sum_{\substack{\text{left}\\\text{handed}\\\text{fermions}}} (q_i)^3 \overset{!}{=} 0$$

$$6\left(\tfrac{1}{3}\right)^3 + 3\left(\tfrac{-4}{3}\right)^3 + 3\left(\tfrac{2}{3}\right)^3 + 2(-1)^3 + (2)^3 = 0 \checkmark$$

If this were not satisfied, $U(1)_Y$ would be sick. Symptom: gauge field would get mass via

Also anomaly with $U(1)_Y$ gauge field & 2 gravitons

$$\text{anomaly} \sim \sum_{\substack{\text{left} \\ \text{handed} \\ \text{fermions}}} q_i \overset{!}{=} 0$$

$$6\left(\tfrac{1}{3}\right) + 3\left(\tfrac{-4}{3}\right) + 3\left(\tfrac{2}{3}\right) - 2 + 2 = 0 \quad ✓$$

Such anomalies are nontrivial in chiral theories - trivial for vectorlike where all q_i come in $\pm$ pairs.