Some group theory:

- Consider $SO(d)$ rotation group in d dimensions

$\vec{X}$ = vector w/ d compts X^i

rotate $X^i \to R^i_j X^j$ preserves $\vec{X}\cdot\vec{X}$

if $R^T R = \mathbb{1}$ orthogonal matrix

$\det R = \pm 1$ take $\det R = +1 \leftarrow SO(d)$

$R = e^{i\phi_a T^a}$ $T^a = (T^a)^\dagger = -(T^a)^T$

T^a = antisymm & pure imaginary

- $\Rightarrow R$ = real & $R^T R = \mathbb{1}$.

$a = 1 \cdots \left(\frac{d(d-1)}{2}\right) \leftarrow$ # antisymm $d \times d$ matrices

Corresponds to angular momentum $M_{ij} = -M_{ji}$

$= X_i P_j - X_j P_i$ = rotation generators

$[T^a, T^b] = i f_{abc} T^c \leftarrow$ Lie algebra

$\uparrow$ structure constants

e.g. $d=3$ $a = 1,2,3$ $M_{ij} = \varepsilon_{ijk} J^k$

- $[J^i, J^j] = i\varepsilon_{ijk} J^k \leftarrow SO(3)$ Lie Alg.

$X^i \to R^i_j X^j$ is the "vector" rep.

other reps from other matrix realizations of

$[T^a, T^b] = i f^{abc} T^c$.

E.g. "adjoint" rep: $(T^a)^{bc} = f^{abc}$

dimension $= \dim(G) = \frac{d(d-1)}{2}$ here

This is the antisymmetric rep for $SO(d)$

$X^{[ij]} \longrightarrow R^{[i}_{k} R^{j]}_{\ell} X^{[k\ell]}$

$$\underset{\substack{\text{vector} \times \text{vector} \\ \vec{X} \quad \vec{y}}}{d \times d} = \underbrace{(1)}_{\text{singlet} = \vec{x}\cdot\vec{y}} + \underbrace{\left(\frac{d(d-1)}{2}\right)}_{\substack{\text{antisymm} = \text{adj for} \\ SO(d)}} + \left(\frac{d(d+1)}{2} - 1\right) \leftarrow \text{symm traceless rep}$$

<u>"Little group"</u>: Subgroup which preserves e.g. some constant nonzero vector $\vec{X}_0$. Take $\vec{X}_0 = (c, 0, 0, 0)$

$SO(d-1) \subset SO(d)$ preserves $\vec{X}_0$, so little group of a vector $= SO(d-1)$.

Invariant tensors: δ^i_j, $\varepsilon^{i_1 \cdots i_d}$

Consider moving up to $(1, d-1)$ dimensions (Minkowski) or d dimensions Euclidean.

Poincare group = symmetries of ds^2

translations, rotations, boosts. Consider Lorentz subgroup of rotations & boosts (no translations).

Lorentz group for Minkowski signature preserves $t^2 - \vec{x}^2$ = rotation group $SO(1, d-1)$.

For Euclidean case preserves $\tau^2 + \vec{x}^2$ = rotation group $SO(d)$.

E.g. 1+1 Minkowski: $\begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}$ boosts

vs 2d Euclidean $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotation.

1+1 boosts : $SO(1,1)$ vs 2d rotations $SO(2)$.

$SO(2)$ rotates around a circle: compact gp.

$SO(1,1)$ boosts along hyperboloid $(t^2 - x^2) \rightarrow$ noncompact group.

Generally $SO(d)$ = compact

$SO(1, d-1)$ = noncompact $\leftrightarrow$ contains $SO(1,1)$ as a subgp.

Terminology: since we want to include fermions = spinor representations, the names of the groups we want are technically Spin(d) or Spin$(1,d-1)$.

Spin(d) & Spin$(1,d-1)$ are similar but have important differences. Consider e.g. $d=3$:

Mink: $\mathrm{Spin}(1,2) \simeq SL(2,\mathbb{R}) \simeq SU(1,1)$

Eucl: $\mathrm{Spin}(3) \simeq SU(2)$.

Review SU(2): $X^\alpha \equiv$ 2d complex vector $(\alpha = 1,2)$ (4 real compts)

Symmetries: $\mathcal{U} : X^\alpha \to \mathcal{U}^\alpha_\beta X^\beta$ (sum repeated $\alpha,\beta = 1,2$)

$\mathcal{U}^\alpha_\beta =$ Unitary 2×2 matrix with $\det \mathcal{U} = 1$

e.g. $\mathcal{U} = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$
a, b complex.

Writing $a = a_1 + i a_2$ $b = b_1 + i b_2$, the group elements satisfy $(a_1)^2 + (a_2)^2 + (b_1)^2 + (b_2)^2 = 1$ unit S^3 in $\mathbb{C}^2 \simeq \mathbb{R}^4$. $SU(2) \simeq S^3$ in same sense that $SO(2)$ $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \simeq S^1$ ($S^1 \equiv$ circle).

Let also $\mathcal{U} : X_\alpha \to \mathcal{U}^{*}{}_\alpha{}^\beta X_\beta$

So $X_\alpha X^\alpha =$ inv't by $\mathcal{U}^\dagger \mathcal{U} = \mathbb{1}$

Also define $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$ & $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$

with $\varepsilon^{12} = -\varepsilon_{12} = +1$.

$$\left.\begin{aligned} \mathcal{U}: \quad \varepsilon^{\alpha\beta} &\to \det\mathcal{U}\ \varepsilon^{\alpha\beta} = \varepsilon^{\alpha\beta} \\ \varepsilon_{\alpha\beta} &\to \det\mathcal{U}\ \varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta} \end{aligned}\right\} \text{invt.}$$

Use $\varepsilon^{\alpha\beta}$ & $\varepsilon_{\alpha\beta}$ to raise & lower indices

$X_\alpha = \varepsilon_{\alpha\beta} X^\beta$, $\quad X^\alpha = \varepsilon^{\alpha\beta} X_\beta \qquad \left(\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma\right)$

$$\left.\begin{aligned} X^\alpha &\to 2 \text{ of } SU(2) \\ X_\alpha &\to \bar{2} \text{ of } SU(2) \end{aligned}\right\}$$

Using $\varepsilon^{\alpha\beta}$ & $\varepsilon_{\alpha\beta}$

$2 \simeq \bar{2} \leftarrow$ Special prop. of $SU(2)$.

$2 \simeq \bar{2} \longrightarrow$ rep. 2 is "pseudoreal".

X^α has 2 complex $= 4$ real compts. Can we impose a reality condition $X_\alpha^* = X_\alpha$ or $X_\alpha^* = \varepsilon_{\alpha\beta} X_\beta$? No – not compatible with $X^\alpha \to \mathcal{U}^\alpha_\beta X^\beta$ (unless $X^\alpha = 0$). So spinor of $Spin(3) \simeq SU(2)$ has a minimum of 2 complex $= 4$ real components.

$2 \times 2 = (1)_A + (3)_S \quad$ or $\quad \frac{1}{2} \times \frac{1}{2} = (0)_A + (1)_S$

$\uparrow$ label by dim of rep $= 2j+1$

label by spin j of rep.

$|j, m\rangle \quad m = -j \cdots j$

Spin $j=0$ state is precisely

$$\varepsilon_{\alpha\beta}\, x^\alpha y^\beta \longrightarrow |+-\rangle - |-+\rangle$$

Spin $j=1$ states are $\sigma^i_{\alpha\beta}\, x^\alpha y^\beta$

$$\sigma^i_{\alpha\beta} = \sigma^i_{\beta\alpha} \qquad i = 1,2,3 \qquad \left\{ \begin{array}{l} |{+}{+}\rangle \\ |+-\rangle + |-+\rangle \\ |--\rangle \end{array} \right.$$

$(\sigma^i)_\alpha{}^\beta$ = Pauli matrices $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ = basis of traceless Hermitian matrices

entering in $(U)^\alpha_\beta = \left(e^{\frac{i}{2}\vec{\phi}\cdot\vec{\sigma}}\right)^\alpha_\beta$.

$\sigma^i_{\alpha\beta} = \varepsilon_{\beta\gamma}(\sigma^i)_\alpha{}^\gamma$ = symmetric matrices.

Let e.g. $P_\alpha{}^\beta = \vec{\sigma}_\alpha{}^\beta \cdot \vec{P} = \begin{pmatrix} P_z & P_x - iP_y \\ P_x + iP_y & -P_z \end{pmatrix}$

$$\det(P_\alpha{}^\beta) = -(P_x^2 + P_y^2 + P_z^2) = -\vec{P}\cdot\vec{P}.$$

Now consider $\mathrm{Spin}(1,2) \simeq SU(1,1) \simeq SL(2,\mathbb{R})$

e.g. $SL(2,\mathbb{R})$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad a,b,c,d$ all real, $ad - bc = 1$.

Still 3 real dim'l but now not S^3, instead noncompact.

Spinor $\chi^\alpha \to \mathcal{U}^\alpha{}_\beta \chi^\beta$ with $\mathcal{U}^\alpha_\beta = SL(2,\mathbb{R})$ matrix.

○ Now no problem to take $\chi^\alpha =$ real. $\alpha = 1,2$

So whereas spin(3) spinor = 4 real dim'l, spin (1,2) = 2 real dim'l. This is referred to as a Majorana spinor in 1+2 dimensions.

Still use $\varepsilon_{\alpha\beta}$ & $\varepsilon^{\alpha\beta}$ = invt. (since $\det \mathcal{U} = 1$) to raise & lower indices.

$\exists$ 1+2 dim'l Minkowski susy algebra with 2 real supercharges $Q_\alpha = (Q_\alpha)^\dagger$

○

$$\{Q_\alpha, Q_\beta\} = 2\,\Gamma^\mu_{\alpha\beta}\, P_\mu$$

$$\Gamma^\mu_{\alpha\beta} = \varepsilon_{\beta\gamma} (\Gamma^\mu)_\alpha{}^\gamma \qquad \left.\begin{aligned} (\Gamma^0)_\alpha{}^\gamma &= (i\sigma_2)_\alpha{}^\gamma \\ (\Gamma^1)_\alpha{}^\gamma &= (\sigma_1)_\alpha{}^\gamma \\ (\Gamma^2)_\alpha{}^\gamma &= (\sigma_3)_\alpha{}^\gamma \end{aligned}\right\} \text{all real}$$

$\Gamma^\mu_{\alpha\gamma}$ = real & symmetric.

See appendix B of Vol 2 Polchinski for general discussion of Spin (1,d-1) & Spin(d)

○ spinor reps. Idea: find Γ^μ Dirac matrices s.t. $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$

For $d = 2k$ the "Dirac" rep is 2^k dim'l

Work in $\mathbb{R}^{2k} \to \mathbb{C}^k$ with fermion creation & annihilation ops ψ_I, ψ_I^* $I = 1 \cdots k$

Take $|\Omega\rangle$ s.t. all $\psi_I |\Omega\rangle = 0$. Use ψ_I^* to fill out rep.

$|\Omega\rangle$, $\psi_I^* |\Omega\rangle$, $\psi_I^* \psi_J^* |\Omega\rangle$, ...

$$\dim = \sum_{n=0}^{k} \binom{k}{n} = 2^k$$

$(k = d/2)$

each ψ_I^* can act or not act on $|\Omega\rangle$ $\hookrightarrow$ 2^k states.

This is 2^k complex compts $= 2^{k+1}$ real compts.

Generally reducible. Can impose Weyl cond for eigenvalue of $\Gamma \equiv (i)^{-(k-1)} \Gamma_0 \cdots \Gamma_{d-1}$

$\Gamma = +1 \longrightarrow$ one chirality rep

$\Gamma = -1 \longrightarrow$ another chirality rep

like keeping only $(-1)^F = +1$ states or only $(-1)^F = -1$ states in

$\psi_{I_1}^* \cdots \psi_{I_n}^* |\Omega\rangle. \longrightarrow 2^k$ real dim'l rep.

$\Gamma = 1$: even # ψ_I^* / $\Gamma = -1$: odd # ψ_I^*