Consider $S = \int dt\, d\theta\, d\theta^* \left(-\frac{1}{2} D\Phi\, D^+\Phi + W(\Phi) \right)$

$\rightarrow \mathcal{L} = \frac{1}{2}\dot{\phi}^2 + i\psi^+\dot{\psi} - \frac{1}{2}(W')^2 - W''\psi^+\psi$

For $W(\Phi) = \frac{1}{3}\Phi^3 + a\Phi$. Classical vacua at $W'(\phi) = \phi^2 + a = 0$. 2 cases $a>0$ or $a<0$

i) $a>0$ no sol'n to $W'(\phi)=0 \Rightarrow$ susy is classically spont. broken. Vacuum at $\langle\phi\rangle = 0$

$W''(\phi)\psi^+\psi \longrightarrow$ massless fermion since $W''(\langle\phi\rangle) = 0$. This is a general property:

$\boxed{\text{Global susy = spont. broken} \Rightarrow \exists \text{ massless fermion}}$

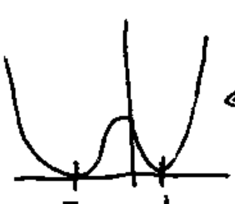
"Goldstino" by analog of Goldstone's Thm.

ii) $a<0$ Classical susy vacua @ $\langle\phi\rangle_\pm = \pm\sqrt{-a}$. These vacua here $(-1)^F = \pm 1$.

Classically susy is unbroken. Also $W''(\phi)\psi^+\psi$ now gives ψ a mass for $\langle\phi\rangle \neq 0$, again susy not broken classically since classically $\nexists$ massless fermion.

In quantum thy: instantons lift energy above zero $\longrightarrow$ susy broken spontaneously via inst. effects. Instanton tunnelling also $\Longrightarrow$ $\langle\phi\rangle = 0$ in true vacuum. So again see $W''(\langle\phi\rangle) = 0$ in vacuum $\Longrightarrow$ ψ a massless Goldstino ✓

We'll soon consider this theory in higher dimensions. Above 0+1 dimensions will find instanton doesn't have finite action (for infinite volume) $\Longrightarrow$ theory with $a>0$ still classically & quantumly has spont. susy breaking but theory with $a<0$ has two susy vacua at $\langle\phi\rangle = \pm\sqrt{-a}$ with no massless fermion. Even though $\mathrm{Tr}(-1)^F = 1-1=0$, susy is unbroken.



$\longleftarrow$ inst. tunnelling only in 0+1 dims

$\phi \to -\phi$ symm spont. broken above 0+1

Susy vacua $\longleftrightarrow$ math connections (Witten)

$Q^2 = (Q^+)^2 = 0 \qquad \{Q, Q^+\} = 2H.$

If $\chi = Q\psi \longrightarrow Q\chi = 0 \qquad (Q^2=0)$

Q: Suppose $Q\chi = 0$. Does this imply $\chi = Q\psi$?

A: If and only if χ is not a susy groundstate.

Pf: If $\chi \neq$ susy groundstate, energy $E \neq 0$

let $\psi \equiv \frac{1}{2E} Q^+\chi$. Then $\chi = Q\psi$

$(Q\psi = \frac{1}{2E} QQ^+ \chi = \chi)$. Now prove other direction: If $E_\chi = 0$, $\chi \neq Q\psi$ since this would imply $E_\psi = 0$, but $E_\psi = 0 \Rightarrow Q\psi = 0$.

So susy groundstates are precisely the states

s.t. $\quad Q\chi = 0 \quad$ ("Q closed")

but not $\chi = Q\psi \quad$ (not "Q exact")

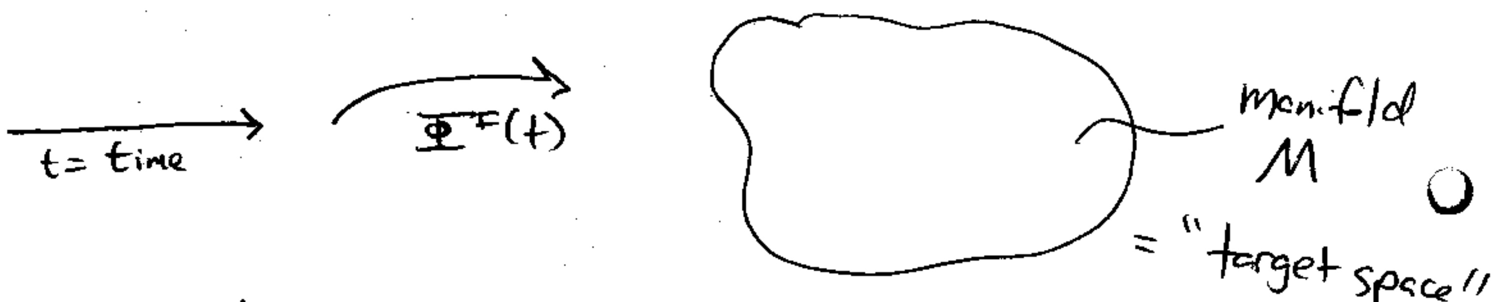
Susy groundstates $\longleftrightarrow \ker Q / \operatorname{im} Q$.

Consider $S = \int dt\, d\theta\, d\theta^* \left[-\frac{1}{2} g_{IJ}(\Phi)\, D\Phi^I D^\dagger \Phi^J + W(\Phi) \right]$. For now take $W(\Phi) = 0$

"Non linear sigma model". Do θ integrals

$$\mathcal{L} \overset{*}{=} \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + \frac{1}{2} g_{IJ} \psi^{*I} \frac{D}{Dt} \psi^J + \frac{1}{8} R_{IJKL}\, \psi^{*I} \psi^J \psi^{*K} \psi^L$$

$*$: extra credit



$g_{IJ}(\phi)$ = metric of manifold M

$R_{IJKL}(\phi)$ = Rieman tensor curvature of M

$$Q = i \sum_I \psi^{*I} p_I \qquad Q^\dagger = -i \sum_I \psi^I p_I$$

$$p_I = -i \frac{D}{D\phi^I} \leftarrow \text{covariant derivative on } M$$

$$\{\psi^{*I}, \psi^J\} = g^{IJ}(\phi) \qquad \text{Clifford alg.}$$

The susy vacua are of form

$$F_{I_1 \cdots I_p}(\phi)\, \psi^{*I_1} \cdots \psi^{*I_p} |\Omega\rangle$$

↑ state annihilated by all ψ^I.

$$Q = \sum_I \psi^{*I} \frac{D}{D\phi^I}$$

Consider $\psi^{*I} \sim d\phi^I$ = basis of 1 forms

Then above state is a p-form and

$Q = d$. Groundstates are d closed mod d exact = cohomology of manifold M!

Also $Q^* = d^* \equiv *d*$. So

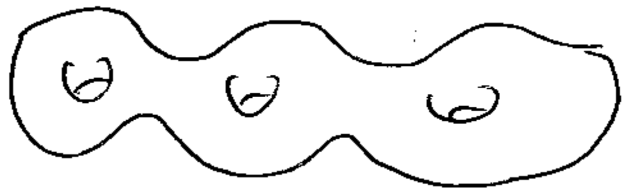
$$2H = dd^* + d^*d = 2\Delta \leftarrow \text{Laplacian on } M.$$

The number of ~susy groundstates with fermion # p = b_p = $\dim \underbrace{H^p(M)}_{\text{space of p-form cohomology elements.}}$ = "p-th Betti number".

$$\boxed{\mathrm{Tr}\,(-1)^F = \sum_p (-1)^p b_p = \text{Euler Character!}}$$

This shows that $\mathrm{Tr}(-1)^F$ is invariant under smooth deformations of metric $g_{IJ}(\phi)$ as expected. The Euler character is a topological invariant of M.

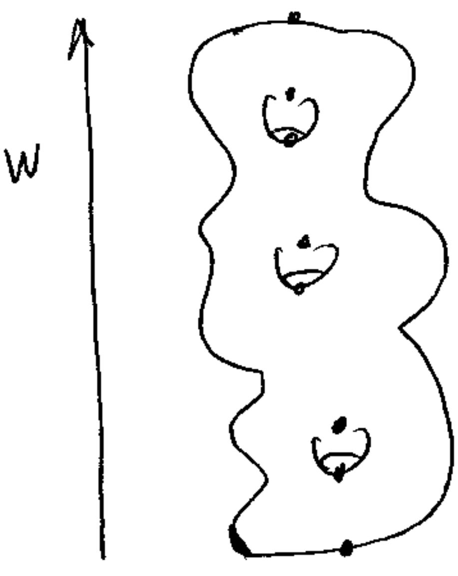
E.g. $M =$



$\leftarrow h$ handles

$$\mathrm{Tr}(-1)^F = b_0 - b_1 + b_2 = 2-2h \equiv \chi$$

$b_0 = b_2 = 1$, $b_1 = 2h$. Another way to see this: Add $W(\Phi) =$ height fn



Critical points of $W(\Phi)$:

top: W'' has 2 negative eigenvals

bottom: W'' has 2 positive eigenvals

Each hole: two places (top & bottom of hole) where $W' = 0$

Each has 1 positive & 1 negative eigenvalue.

$$\mathrm{Tr}(-1)^F = \underbrace{(-1)^2}_{\text{top}} + \underbrace{(1)^2}_{\text{bottom}} + \underbrace{2h(1)(-1)}_{\text{holes}} = 2-2h \checkmark$$

$\boxed{\text{Morse thy}}$