Week 3 Homework (Due Oct. 10)

1. Verify $\{Q, D\} = \{Q^\dagger, D\} = \{D, D\} = 0$ and $\{D, D^\dagger\} = -2H$ using $Q = \frac{\partial}{\partial\theta} + i\theta^* \frac{\partial}{\partial t}$, $Q^\dagger = \frac{\partial}{\partial\theta^*} + i\theta \frac{\partial}{\partial t}$, $D = \frac{\partial}{\partial\theta} - i\theta^* \frac{\partial}{\partial t}$, $D = \frac{\partial}{\partial\theta^*} - i\theta \frac{\partial}{\partial t}$.

2. Consider a theory with N real superfields $\Phi_I = \phi_I + \theta\psi_I - \theta^*\psi_I^* + \theta\theta^* F_I$ $(I = 1 \ldots N)$. Suppose that the action is

$$S = \int dt d\theta d\theta^* \left(-\tfrac{1}{2} \sum_{I=1}^{N} D\Phi_I D^\dagger \Phi_I + W(\Phi) \right),$$

where $W(\Phi)$ is a general function of all Φ_I. Do the $d\theta d\theta^*$ integrals and write out the Lagrangian in terms of the fields ϕ_I, ψ_I, and ψ_I^*, with all F_I eliminated by their equations of motion. What are the conditions for the classical supersymmetric vacua, with fermions set to zero (the generalization of $W' = 0$ to this multi-field case)?

3. In the above N field theory, let $|\Omega\rangle$ be a state which is annihilated by all ψ_I, with the ψ_I^* acting on $|\Omega\rangle$ as creation operators. Let $|\Omega\rangle$ have $(-1)^F$ eigenvalue $+1$ and recall that $\{(-1)^F, \psi_I^*\} = 0$. In this notation, we write the groundstate for the case with $N = 1$ field and $W(\Phi) = m\Phi^2$ as

$$\begin{cases} e^{-W(\phi)/\hbar}|\Omega\rangle & \text{if } m > 0 \\ e^{W(\phi)/\hbar}\psi^*|\Omega\rangle & \text{if } m < 0. \end{cases}$$

So the groundstate has $(-1)^F$ =sign(m). Now consider the N variable case, with $W = \sum_{I=1}^{N} m_I \Phi_I^2$. Suppose that $m_1 \ldots m_K$ are all negative and $m_{K+1} \ldots m_N$ are all positive.

a) How many ground states are there?

b) Write the groundstate(s) in the above form (a function of the ϕ_I combined with the fermion part of the wavefunction, written in terms of the basis generated by $|\Omega\rangle$ via the creation operators ψ_I^*). (Hint: the bosonic part should be familiar: remember your quantum mechanics.)

c) What is the total $\mathrm{Tr}(-1)^F$ of this theory, including the sign?

4. Consider the theory with two superfields, Φ_1 and Φ_2, and $W = \Phi_1(\Phi_2^2 - 1)$.

a) What are the classical vacua?

b) In each classical vacuum, consider the mass matrix for the fluctuations away from the potential minimum. Using the result of the previous problem, find the value of $(-1)^F$ for each classical vacuum.

c) What is the total $\mathrm{Tr}(-1)^F$ of this theory, including the sign?

Last time: $\delta \chi = [\varepsilon^* Q + \varepsilon Q^+, \chi]$

$$\Phi = \phi + \theta\psi - \theta^*\psi^* + \theta\theta^* F$$

$$Q = \frac{\partial}{\partial\theta} + i\theta^*\frac{\partial}{\partial t} \qquad Q^+ = \frac{\partial}{\partial\theta^*} + i\theta\frac{\partial}{\partial t}$$

$$\delta F = i\frac{d}{dt}(\varepsilon\psi^* + \varepsilon^*\psi)$$

so $\int dt\, d\theta\, d\theta^*\, \Phi$ is susy invt $\delta(\,) = 0$

likewise $\int dt\, d\theta\, d\theta^*\, W(\Phi)$ is susy invt.

Use to construct supersymmetric action.

Need also covariant superderivaties

$$D = \frac{\partial}{\partial\theta} - i\theta^*\frac{\partial}{\partial t} \qquad D^+ = \frac{\partial}{\partial\theta^*} - i\theta\frac{\partial}{\partial t}$$

* Verify: $\{D, Q\} = \{D, Q^+\} = \{D, D\} = 0$

$$\{D, D^+\} = -2H$$

<u>Susy action</u>:

$$S = \int dt\, d\theta\, d\theta^* \left(-\tfrac{1}{2} D\Phi D^+\Phi + W(\Phi)\right)$$

This gives $S = \int dt \left(\frac{1}{2}\left(F^2 + \dot{\phi}^2 + i(\bar{\Psi}\dot{\Psi} - \dot{\bar{\Psi}}\Psi)\right) - W'F + \frac{1}{2} W''(\bar{\Psi}\Psi - \Psi\bar{\Psi})\right)$

$(\bar{\Psi} \equiv \psi^\dagger \equiv \psi^*)$

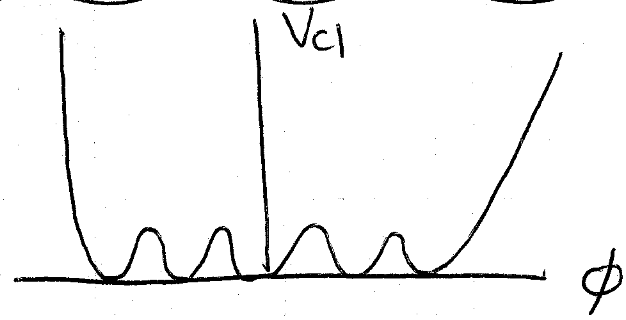
No time derivatives for F. F = auxiliary field which can be solved for once & for all via its E.O.M. $F = W'(\phi)$

$$\Rightarrow S = \int dt \left(\frac{1}{2}\dot{\phi}^2 + i\psi^\dagger\dot{\psi} - \frac{1}{2}(W')^2 + \frac{1}{2}W''[\psi^\dagger, \psi] \right)$$

which gives our $H = \frac{\Pi^2}{2} + \frac{(W')^2}{2} - \frac{1}{2}W''[\psi^\dagger, \psi]$.

Easy to construct susy invariant theories via superspace.

Now back to



only 1 unique $E=0$ vacuum with odd # of wells & no $E=0$ vacua w/ even #.

This happens via tunnelling, which lifts the classical degeneracy. In path integral description of QM or QFT, this process is known as an "<u>instanton</u>" = tunnelling process, order $e^{-c/\hbar}$ = nonperturbative (essential singularity @ $\hbar = 0$). Instanton is a saddle point of the <u>Euclidean</u> action.

<u>Euclidean</u>: $\tau = -it$ $\quad e^{ikx} \to e^{-\kappa x}$ tunnelling

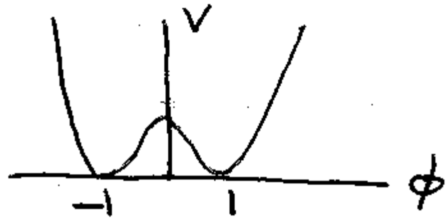
$$e^{\frac{i}{\hbar} S_{Mink}} \longrightarrow e^{-\frac{1}{\hbar} S_{Euc}}.$$

$$S_{Mink} = T - V \longrightarrow S_{Euc} = T + V = H$$

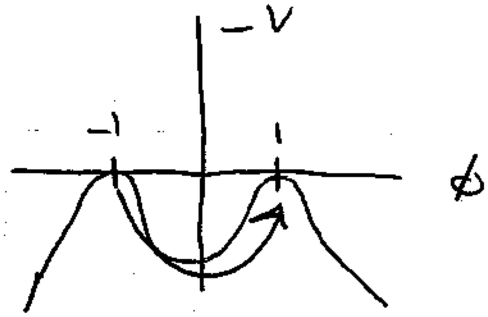
$\int e^{-\frac{1}{\hbar} S_E}$ has saddle points at paths $\phi(t)$ which would be sol'ns of the eqns of motion with potential $V \to -V$.

Concrete example, without fermions for now.

$V = \frac{1}{2}(\phi^2-1)^2$



Instantons are nontrivial Euclidean saddle point sol'ns of



$\phi(\tau=-\infty) = -1$

$\phi(\tau=+\infty) = +1$

Sol'n of $\quad \frac{d^2\phi}{d\tau^2} = \frac{dV}{d\phi} = (\phi^2-1)(2\phi)$

or via energy $\quad 0 = \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 - \frac{1}{2}(\phi^2-1)^2$

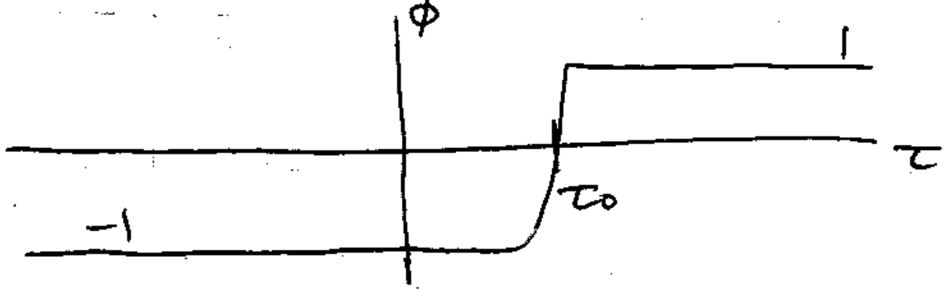
so $\frac{d\phi}{d\tau} = \pm(\phi^2-1)$. Call instanton

case $\quad \phi(-\infty) = -1 \quad \phi(\infty) = +1$ &

anti instanton $\quad \phi(\infty) = -1 \quad \phi(-\infty) = +1$

Instanton sol'n $\quad \phi_a(\tau) = \tanh(\tau - \tau_0)$

The instanton is a kink in Euclidean time localized at arbitrary location τ_0



For correlation fns like $\langle \phi = 1 | e^{-HT/\hbar} | \phi = 1 \rangle$

don't need to tunnel, get amplitude $\sim$

$$\int [d\phi]\, e^{-\frac{1}{\hbar} S[\phi]} = \int \prod d\phi_n\, e^{-\sum \phi_n B_{nm} \phi_m}$$

$$+ O(\hbar) = N (\det B)^{-1/2} + O(\hbar)$$

where $B = -\frac{d^2}{dt^2} + V''(\phi_{cl})$

For tunnelling processes $\langle \phi = 1 | e^{-HT/\hbar} | \phi = -1 \rangle$

saddle pt approx gives amplitude =

$$N e^{-\frac{1}{\hbar} S[\phi_{cl}]} \left[(\det B)^{-1/2} + O(\hbar) \right] \leftarrow$$ not quite right

But find $\det B = 0$ because of zero eigenvalue. 0 eigenvalue = bosonic zero mode = freedom to have arbitrary τ_0

$$\phi_{cl}(\tau - \tau_0 - \delta\tau_0) = \phi_{cl}(\tau-\tau_0) - \delta\tau_0\, \dot{\phi}_{cl}(\tau-\tau_0)$$

$+ \cdots$. Zero mode $\Rightarrow$ $\dot{\phi}_{cl}(\tau-\tau_0)$

How to handle: replace $(\det B)^{-1/2}$ $(=\infty)$

with $(\det' B)^{-1/2} \int d\tau_0 \sqrt{\frac{S(\phi_{cl})}{2\pi\hbar}} \cdots$

normalization factor

$\det' B \equiv$ det with 0 modes omitted, i.e. product of nonzero eigenvalues.

i.e. integrate over instanton zero mode τ_0 = "collective coordinate" Gives factor of

$T = \int d\tau_0$. for $\langle \phi=1 | e^{-HT/\hbar} | \phi=-1 \rangle$

Now let's include the fermions.

Fermionic path integrals $\int [D\bar{\Psi}(t)][D\Psi(t)]\, e^{-S}$

$[D\Psi(t)] \longrightarrow \prod_i d\Psi_i$

$S \rightarrow \sum_{ij} \bar{\Psi}_i F_{ij} \Psi_j \longrightarrow \sum_i \bar{\Psi}_i \lambda_i \Psi_i$

↑ F_{ij} eigenvalues

Get $\prod_i \int d\bar{\Psi}_i d\Psi_i\, e^{-\bar{\Psi}_i \lambda_i \Psi_i}$

Use $e^{-\bar{\Psi}_i \lambda_i \Psi_i} = 1 - \bar{\Psi}_i \lambda_i \Psi_i$

and $\int d\bar{\Psi}_i d\Psi_i\, 1 = 0, \quad \int d\bar{\Psi}_i d\Psi_i\, \bar{\Psi}_i \Psi_i = -1$

so get $\int [d\bar{\Psi}(t)][d\Psi(t)]\, e^{-\int dt\, \bar{\Psi}(t) F(t) \Psi(t)}$

$= \det(F(t)) \qquad F(t) = \text{diff'l op. eg: } \frac{1}{2}\frac{\partial}{\partial t} - \omega''$

<u>Fermion zero modes</u>: suppose more $\bar{\Psi}_i$ than Ψ_j

$F \rightarrow \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_N & \\ & & & 0 \end{pmatrix}$

extra zero modes.

$\int e^{-S} = \det' F \left(\int \prod_i{}' d\bar{\Psi}_i\, 1 \right) = 0$

$\det' F$ = product of nonzero eigenvalues.

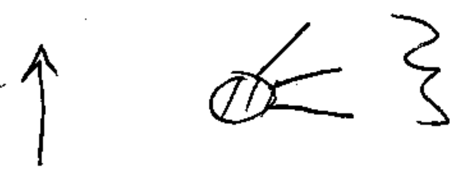
So $\langle 1 \rangle = 0$. Must "soak up zero modes"

$$\langle \prod_i{}' \bar{\psi}_i \rangle = \int e^{-S} \prod_i{}' \bar{\psi}_i = \det' F.$$

So with K zero modes need to consider

$\langle \bar{\psi}(t_1) \cdots \bar{\psi}(t_K) \rangle$ to soak up zero modes

The instantons generally have fermion zero

modes  fermion zero modes.

Supersymmetry actually $\Rightarrow$ there must be some

fermionic zero modes which are superpartners of

bosonic zero modes. Bosonic zero mode

corresponds to spontaneously broken translation

invariance of instanton located at τ_0.

$\boxed{\text{General result, true for any \# of spacetime dimensions \& conserved supercharges: the instanton is annihilated by half of the supercharges and the other half give the fermionic zero modes which are superpartners of the translation zero modes}}$

$\boxed{\text{BPS sol'ns}}$

E.g. for our susy Q.M.:

instanton: Q^{+} annihilates, Q generates zero mode

anti instanton: Q annihilates, Q^{+} generates zero mode

Recall: $Q = \psi(W' + i\pi)/\sqrt{\hbar}$ $\quad Q^{+} = \psi^{+}(W' - i\pi)/\sqrt{\hbar}$

$$\pi = \frac{d\phi}{dt} = i\frac{d\phi}{d\tau}$$

So instanton: Q^{+} annihilates $\Rightarrow \frac{d\phi}{d\tau} = -W'$

anti instanton Q annihilates $\Rightarrow \frac{d\phi}{d\tau} = W'$

These indeed are classical zero energy sol'ns of the Euclidean action (here $\circ \equiv \frac{d}{d\tau}$)

$$\mathcal{L}_E = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(W')^2 - \psi^+\dot{\psi} - W''\psi^+\psi + \cdots$$

take $\psi_{cl} = \psi_{cl}^+ = 0$, $\ddot{\phi}_{cl} - W'W'' = 0$

e.g. $W = \frac{\phi^3}{3} - \phi \Rightarrow V_{cl} = \frac{1}{2}(\phi^2-1)^2$

double well potential considered before.

Instanton $\phi_{cl}(\tau) = \tanh(\tau - \tau_0)$ is sol'n

of $Q^+ = 0 \Rightarrow \frac{d\phi}{d\tau} = -W' = 1-\phi^2$ ✓.

Q generates the fermi zero mode superpartner

to $\frac{\delta}{\delta\tau_0} \rightarrow \dot{\phi}_{cl}(\tau)$

Fermi zero mode $= \eta_0 W' \sim \eta_0 \dot{\phi}$

$\eta_0 =$ constant zero mode fermi spinor (like τ_0)

For instanton / anti-instanton $\quad \phi_{cl} = \mp W'$

$$S_{cl} = \int dt\, \mathcal{L}_E(\phi_{cl}) = \int dt\, W' \dot{\phi}_{cl} = \int dW$$

$$= W(\phi_2) - W(\phi_1) \equiv \Delta W$$

$$S(\phi_{cl} + \delta\phi, \delta\psi, \delta\bar{\psi}) = \int dt \Big[\tfrac{1}{2} (\dot{\phi}_{cl} + \delta\dot{\phi})^2$$

$$+ \tfrac{1}{2} \big(W'(\phi_{cl} + \delta\phi) \big)^2 - \delta\bar{\psi}\, \delta\dot{\psi} - W''(\phi_{cl} + \delta\phi)$$

$$\delta\bar{\psi}\, \delta\psi \Big] = \Delta W + \tfrac{1}{2} \int dt \big(\delta\phi\, B\, \delta\phi + \delta\bar{\psi}\, F\, \delta\psi \big)$$

$$F \equiv \partial_t - W'' \qquad B = -\partial_t^2 + W''' W' + (W'')^2$$

$$= (-\partial_t - W'')(\partial_t - W'')$$

$$= F^+ F$$

So $\displaystyle\int DX\, e^{-S} \simeq e^{-\Delta W/\hbar} (\det{}' B)^{-1/2} (\det{}' F)\, (1+O(\hbar))$

up to zero mode stuff $\rightarrow \int d\tau_0 \int d\eta_0$

Nice susy cancellation: $(\det{}' B)^{-1/2} (\det{}' F) = \underline{1}$!

<u>Reason</u>: $B\xi = \lambda\xi$ for $\lambda \neq 0 \Rightarrow$

$\bar{F}\eta = \sqrt{\lambda}\,\xi$ & $F\xi = \sqrt{\lambda}\,\eta$ for $\eta \equiv \frac{F\xi}{\sqrt{\lambda}}$

So $\forall$ bosonic eigenvector w/ eigenvalue $\lambda \neq 0$
$\exists$ F eigenvector w/ eigenvalue $\sqrt{\lambda}$

$\therefore\ (\det' B)^{-1/2}(\det' F) = 1 \leftarrow$ often happens with susy!

Only need to deal with zero modes,
non zero modes cancel in pairs (boson/fermion).

Fermi zero mode of instanton generated by Q

note $F^{\dagger}\dot{\phi}_{cl} = (\partial_\tau + W'')\dot{\phi}_{cl} = -(\partial_\tau + W'')W'$

$= -W''\dot{\phi}_{cl} - W''W' = 0$

Bosonic zero mode $\delta\phi = \varepsilon_0\,\dot{\phi}_{cl}$

$F^{\dagger}$ fermionic zero mode $\delta\psi = \eta_0\,\dot{\phi}_{cl}$

must insert Q or ψ into amplitude
to soak up $\delta\psi$ zero mode. Set $\psi = \delta\psi = \eta_0\dot{\phi}_{cl}$

E.g. Non-zero energy degenerate groundstates

of $\mathrm{Tr}(-1)^F = 0$ thy $W \sim \phi^K$ K odd:

$|0_+\rangle$ & $|0_-\rangle$

$$E_0 = \langle 0_- | H | 0_- \rangle = \tfrac{1}{2} \langle 0_- | \{Q, Q^+\} | 0_- \rangle$$

$$= \tfrac{1}{2} \langle 0_- | Q^+ Q | 0_- \rangle \simeq \tfrac{1}{2} \left| \langle 0_+ | Q | 0_- \rangle \right|^2$$

$$\langle 0_+ | Q | 0_- \rangle \sim e^{-\Delta W/\hbar} \int d\eta_0 \int d\tau_0$$

$$\Big[\underbrace{\eta_0 \dot\phi_{cl} (\dot\phi \,\bar{}\, W')}_{Q_{cl}} + O(\hbar) \Big]$$

$$\sim e^{-\Delta W/\hbar} \int d\phi \cdot \left(-2 \frac{dW}{d\phi} \right)$$

$$\sim \Delta W e^{-\Delta W/\hbar}$$

$$E_0 \sim e^{-2\Delta W/\hbar} > 0$$

instanton lifts energy, & breaks susy