Example of SUSY QM: $\Omega = \begin{pmatrix} f_+(\phi) \\ f_-(\phi) \end{pmatrix}$

$$\Pi = -i\hbar \frac{\partial}{\partial \phi}.$$

$$Q \equiv \sigma^- \left(W'(\phi) + i\Pi \right) \qquad \sigma^- \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$Q^+ \equiv \sigma^+ \left(W'(\phi) - i\Pi \right) \qquad \sigma^+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$(-1)^F = \sigma^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

* Ex: verify

potential — magnetic field

$$\{Q^+, Q\} \overset{*}{=} \Pi^2 + (W')^2 - \hbar W''(\phi)\sigma^3 \equiv 2H$$

$Q^2 = 0$ & $Q^{+2} = 0$ since $(\sigma^\pm)^2 = 0$

Is there a supersymmetric groundstate?

Need $Q|\Omega\rangle = Q^+|\Omega\rangle = 0$

$$\Omega = \begin{pmatrix} f_+ \\ 0 \end{pmatrix} \overset{*}{\Longrightarrow} (W' + i\Pi) f_+ = 0$$

$$\Longrightarrow f_+ \propto e^{-W/\hbar}$$

$$\Omega = \begin{pmatrix} 0 \\ f_- \end{pmatrix} \overset{*}{\Longrightarrow} (W' - i\Pi) f_- = 0$$

$$\Longrightarrow f_- \propto e^{W/\hbar}$$

* note 1st order not 2nd order diff'l eqns!

Normalizable condition:

(1) $\lim\limits_{\phi \to \pm\infty} W(\phi) \to +\infty \implies$ f_+ normalizable, f_- not

(2) $\lim\limits_{\phi \to \pm\infty} W(\phi) \to -\infty \implies$ f_- normalizable, f_+ not

(3) $\lim\limits_{\phi \to +\infty} W(\phi) = -\lim\limits_{\phi \to -\infty} W(\phi) \implies$ neither normalizable

Cases (1) & (2) : <u>unique</u> susy vacuum

case (3) : susy is broken

E.g. $W(\phi) = \phi^K$: susy is unbroken

with unique susy vacuum, for K even &

susy is broken for K odd. (Explicitly for $K=1$, spontaneously for $K=3,5,7,\ldots$)

$\mathrm{Tr}(-1)^F = 1$ for K even

$\mathrm{Tr}(-1)^F = 0$ for K odd

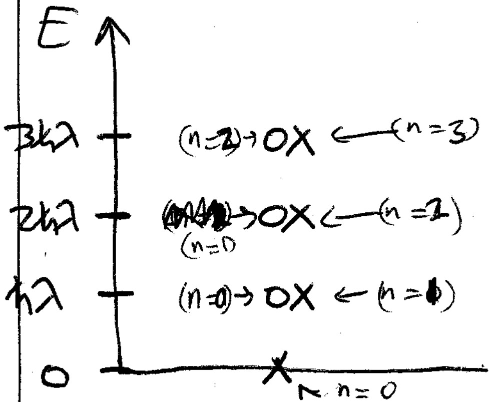
Consider now $W(\phi) = \frac{\lambda}{2}(\phi - a)^2$

$$H = \frac{\Pi^2}{2} + \frac{\lambda^2}{2}(\phi - a)^2 - \frac{\hbar}{2}\lambda\sigma_3$$

Eigenstates $\Omega^n_+ = \begin{pmatrix} \omega_n(\phi) \\ 0 \end{pmatrix}$ $\quad \Omega^n_- = \begin{pmatrix} 0 \\ \omega_n(\phi) \end{pmatrix}$

$\omega_n(\phi) = \langle \phi | n \rangle =$ Harmonic osc. eigenfunctions $\left(= \begin{matrix}\text{Hermite} \\ \text{polynomials}\end{matrix}\right)$

$$H\Omega^n_\pm = E^n_\pm \Omega^n_\pm \qquad E^n_\pm = \hbar\lambda\left(n + \tfrac{1}{2} \mp \tfrac{1}{2}\right)$$



x : $(-1)^F = 1 \longleftarrow E^n_+ = \hbar\lambda n$

o $(-1)^F = -1 \longleftarrow E^n_- = \hbar\lambda(n+1)$

Total $\mathrm{Tr}(-1)^F e^{-\beta H} = +1$

Groundstate $\begin{pmatrix} f_+ \\ 0 \end{pmatrix}$ $\quad f_+ \sim e^{-\frac{\lambda}{2}(\phi - a)^2}$

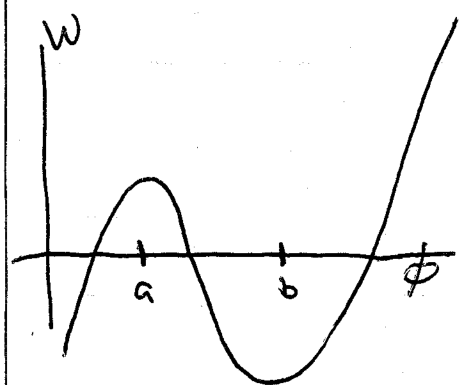
is indeed the harmonic oscillator groundstate wave function. ✓ Again, obtained from 1st order diff'l eqns $\rightarrow$ $f_+ \sim e^{-W/\hbar}$ = completely general & exact sol'n.

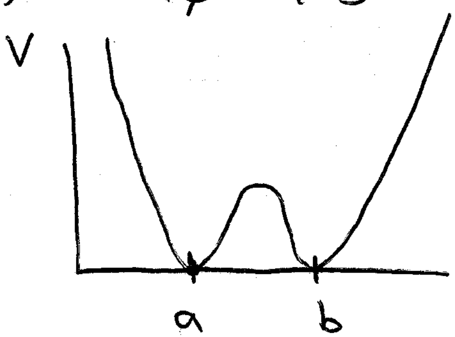
Next example

$$W = \frac{\phi^3}{3} + \alpha\phi^2 + \beta\phi + \gamma \rightarrow W' = (\phi - a)(\phi - b)$$

potential: $V \simeq (W')^2 = (\phi-a)^2(\phi-b)^2$

mag field: $W''(\phi) = 2\phi - a - b$





From exact analysis of susy vacua we know this is case (3) with no normalizable groundstate $\longrightarrow$ no $E=0$ vacuum. But naively V has two $E=0$ vacua at a & b above. Resolution familiar in QM: let $|a\rangle$ & $|b\rangle$ be approx groundstates centered at $\varphi = a, b$.

Unique groundstate $\sim |a\rangle + |b\rangle$ with $|a\rangle - |b\rangle$ higher energy. Recall in QM

true groundstate has no nodes in wavefn &
is unique. Mixed states $|a\rangle + |b\rangle$ &
$|a\rangle - |b\rangle$ both have non-zero energy.

Energy $\sim e^{-c/\hbar}$ nonperturbative effect
associated with tunnelling between opposit.
vacuum. This is an example of "dynamical
spont. susy breaking due to instantons" - more
details soon.

Witten index for $W = \lambda\phi^3 + g\phi^2$

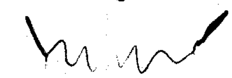
$$\mathrm{Tr}(-1)^F e^{-\beta H} = 0 \quad \text{if } \lambda \neq 0, \; g \text{ arbitrary}$$

$$\mathrm{Tr}(-1)^F e^{-\beta H} = 1 \quad \text{if } \lambda = 0, \; g \neq 0$$

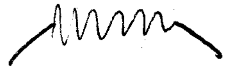
Makes sense: for $\lambda \neq 0$, changing g is a smooth
variation $\rightarrow$ $\mathrm{Tr}(-1)^F$ invariant. Taking $\lambda \rightarrow 0$

Last time: 3 cases

a) $W(\pm\infty) = +\infty \Rightarrow \Omega = \begin{pmatrix} e^{-W/\hbar} \\ 0 \end{pmatrix}$ unique susy groundstate $\mathrm{Tr}(-1)^F = +1$

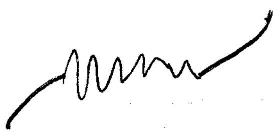


b) $W(\pm\infty) = -\infty \Rightarrow \Omega = \begin{pmatrix} 0 \\ e^{W/\hbar} \end{pmatrix}$ unique susy groundstate $\mathrm{Tr}(-1)^F = -1$



c) $W(\pm\infty) = \pm\infty \Rightarrow$ no susy vacuum. Susy broken

$\mathrm{Tr}(-1)^F = 0$



<u>So</u>

$$W = \phi^K \longrightarrow \mathrm{Tr}(-1)^F = \begin{cases} 1 & K \text{ even} \\ 0 & K \text{ odd} \end{cases}$$

Invariant Witten index under smooth perturbations:

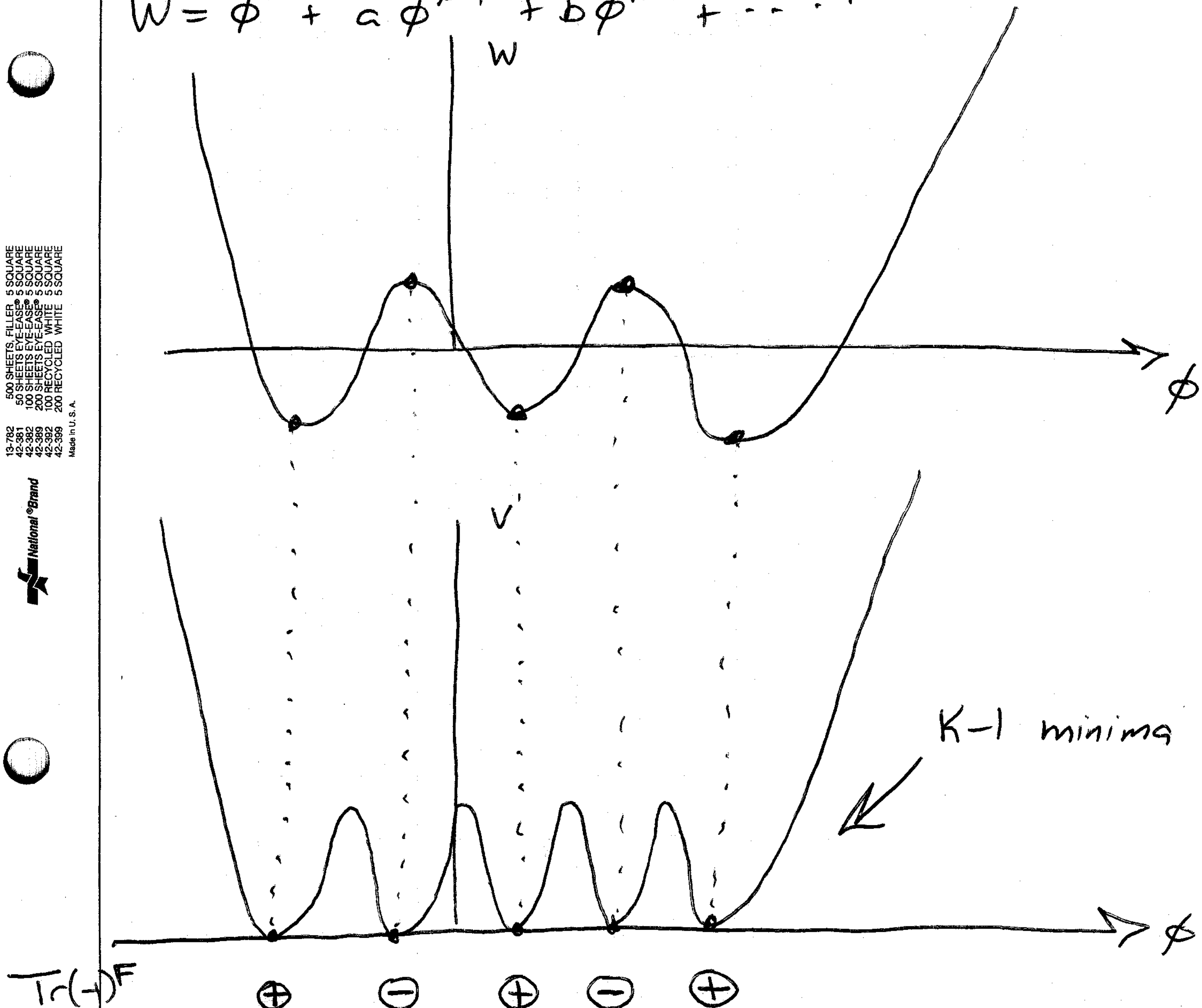
$$W = \lambda \phi^K + a\phi^{K-1} + b\phi^{K-2} + c\phi^{K-3} + \cdots$$

For $\lambda \neq 0$, $\mathrm{Tr}(-1)^F$ indep of $a, b, c, \ldots$

For $\lambda = 0$ $\mathrm{Tr}(-1)^F$ changes, e.g. for K even $\mathrm{Tr}(-1)^F$ goes from 1 to 0 (for $a \neq 0$)

$\lambda = 0$ singular point.

$$W = \phi^K + a\phi^{K-1} + b\phi^{K-2} + \cdots$$



Exact groundstate $\Omega = \begin{pmatrix} e^{-W/\hbar} \\ 0 \end{pmatrix}$ peaked at ⊕ vacua

See classically $\mathrm{Tr}(-1)^F = \begin{cases} 1 & K \text{ even} \\ 0 & K \text{ odd} \end{cases}$

Exact answer since $\mathrm{Tr}(-1)^F$ inv't under deforming th.

Tunnelling $\Longrightarrow$ ⊕ ⊖ pairs get lifted.

Is a singular variation & $Tr(-1)^F$ can change under such variations via susy vacua moving in or out from infinity. Here, for $\lambda \to 0$, instanton mixing goes away restores supersymmetry.

Take as another example $W = \lambda x^4 + x^3$

$Tr(-1)^F = 1$ for $\lambda \neq 0$ & $Tr(-1)^F = 0$ for $\lambda = 0$

Susy vacuum at $x = -1/\lambda$ runs out to ∞ for $\lambda \to 0$.

<u>Superspace</u> $\psi \equiv \sqrt{\hbar}\,\sigma^+ \qquad \psi^+ \equiv \sqrt{\hbar}\,\sigma^-$

$$\{\psi, \psi\} = \{\psi^+, \psi^+\} = 0 \qquad \{\psi^+, \psi\} = \hbar$$

$$Q = \psi(W' + i\pi)/\sqrt{\hbar} \qquad Q^+ = \psi^+(W' - i\pi)/\sqrt{\hbar}$$

$\{Q^+, Q\} = 2H$ with $H = \frac{\pi^2}{2} + \frac{(W')^2}{2} - \frac{1}{2}[\psi^+, \psi]W''$

derive H from $S = \int dt \left(\frac{1}{2}\dot{\phi}^2 + i\psi^+\dot{\psi} - \frac{1}{2}(W')^2 + \frac{1}{2}W''[\psi^+, \psi] \right)$

* Verify S is real (w/ convention
$(\theta_1\theta_2)^* = \theta_2^* \theta_1^*$ for anticommuting θ's)

Susy Variations $\delta\chi = [\varepsilon^* Q + \varepsilon Q^\dagger, \chi]$

* Verify $\delta\phi = \varepsilon^* \psi - \varepsilon \psi^\dagger$ $(\hbar = 1)$
$\delta\psi = \varepsilon(W' - i\pi)$

Superspace: $H = i\partial_t$ generalize for Q & $Q^\dagger$
via anticommuting coordinates θ & θ^*, ($0 = \theta^2 = \theta^{*2}$
$\theta\theta^* = -\theta^*\theta$).

$$\Phi(t, \theta, \theta^*) = \phi(t) + \theta\psi(t) - \theta^*\psi^*(t) + \theta\theta^* F(t)$$

satisfies $\Phi^* = \Phi$ for ϕ & F real.

$$Q = \frac{\partial}{\partial\theta} + i\theta^* \frac{\partial}{\partial t} \qquad Q^\dagger = \frac{\partial}{\partial\theta^*} + i\theta\frac{\partial}{\partial t}$$

$$\left(\left(\frac{\partial}{\partial\theta}\right)^\dagger = \frac{\partial}{\partial\theta^*} \quad \& \quad \left(i\frac{\partial}{\partial t}\right)^\dagger = i\frac{\partial}{\partial t} \right)$$

satisfy $\{Q, Q^\dagger\} = 2i\partial_t = 2H$ ✓
$\{Q, Q\} = 0$

$$[Q, \Phi] = \left(\frac{\partial}{\partial\theta} + i\theta^* \frac{\partial}{\partial t}\right)\Phi \overset{*}{=} \psi + \theta^*(F + i\dot\phi)$$

$$-i\theta\theta^* \dot\psi \quad \overset{*}{\Longrightarrow} \quad \begin{aligned} \delta\phi &= \varepsilon^*\psi - \varepsilon\psi^* \\ \delta\psi &= \varepsilon(F - i\dot\phi) \\ \delta F &= -i(\varepsilon^*\dot\psi - \varepsilon\dot\psi^*) \end{aligned}$$

($*$ = verify these)

$\delta F = \frac{d}{dt}(\ldots)$ so $\delta\left(\int dt\, d\theta\, d\theta^*\, W(\Phi)\right) = 0$

Need also covariant superderivatives

$$D = \frac{\partial}{\partial\theta} - i\theta^* \frac{\partial}{\partial t} \qquad D^\dagger = \frac{\partial}{\partial\theta^*} - i\theta \frac{\partial}{\partial t}$$

$$\left.\begin{aligned} \{D, Q\} = \{D, Q^\dagger\} = \{D, D\} = 0 \\ \{D, D^\dagger\} = -2H \end{aligned}\right\} * \text{ Verify}$$

$$S = \int dt\, d\theta\, d\theta^* \left(-\tfrac{1}{2} D\Phi\, D^\dagger\Phi + W(\Phi)\right)$$

= most general action w/o higher derivative terms