Week 3 Homework (Due Oct. 10)

1. Verify $\{Q, D\} = \{Q^\dagger, D\} = \{D, D\} = 0$ and $\{D, D^\dagger\} = -2H$ using $Q = \frac{\partial}{\partial\theta} + i\theta^* \frac{\partial}{\partial t}$, $Q^\dagger = \frac{\partial}{\partial\theta^*} + i\theta \frac{\partial}{\partial t}$, $D = \frac{\partial}{\partial\theta} - i\theta^* \frac{\partial}{\partial t}$, $D = \frac{\partial}{\partial\theta^*} - i\theta \frac{\partial}{\partial t}$.

2. Consider a theory with N real superfields $\Phi_I = \phi_I + \theta\psi_I - \theta^*\psi_I^* + \theta\theta^* F_I$ $(I = 1 \ldots N)$. Suppose that the action is

$$S = \int dt d\theta d\theta^* \left(-\tfrac{1}{2} \sum_{I=1}^{N} D\Phi_I D^\dagger \Phi_I + W(\Phi) \right),$$

where $W(\Phi)$ is a general function of all Φ_I. Do the $d\theta d\theta^*$ integrals and write out the Lagrangian in terms of the fields ϕ_I, ψ_I, and ψ_I^*, with all F_I eliminated by their equations of motion. What are the conditions for the classical supersymmetric vacua, with fermions set to zero (the generalization of $W' = 0$ to this multi-field case)?

3. In the above N field theory, let $|\Omega\rangle$ be a state which is annihilated by all ψ_I, with the ψ_I^* acting on $|\Omega\rangle$ as creation operators. Let $|\Omega\rangle$ have $(-1)^F$ eigenvalue $+1$ and recall that $\{(-1)^F, \psi_I^*\} = 0$. In this notation, we write the groundstate for the case with $N = 1$ field and $W(\Phi) = m\Phi^2$ as

$$\begin{cases} e^{-W(\phi)/\hbar}|\Omega\rangle & \text{if } m > 0 \\ e^{W(\phi)/\hbar}\psi^*|\Omega\rangle & \text{if } m < 0. \end{cases}$$

So the groundstate has $(-1)^F$ =sign(m). Now consider the N variable case, with $W = \sum_{I=1}^{N} m_I \Phi_I^2$. Suppose that $m_1 \ldots m_K$ are all negative and $m_{K+1} \ldots m_N$ are all positive.

a) How many ground states are there?

b) Write the groundstate(s) in the above form (a function of the ϕ_I combined with the fermion part of the wavefunction, written in terms of the basis generated by $|\Omega\rangle$ via the creation operators ψ_I^*). (Hint: the bosonic part should be familiar: remember your quantum mechanics.)

c) What is the total $\mathrm{Tr}(-1)^F$ of this theory, including the sign?

4. Consider the theory with two superfields, Φ_1 and Φ_2, and $W = \Phi_1(\Phi_2^2 - 1)$.

a) What are the classical vacua?

b) In each classical vacuum, consider the mass matrix for the fluctuations away from the potential minimum. Using the result of the previous problem, find the value of $(-1)^F$ for each classical vacuum.

c) What is the total $\mathrm{Tr}(-1)^F$ of this theory, including the sign?