For our general closed string action w/ massless background fields

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\det h} \left(\left(h^{ab} G_{\mu\nu} + i\epsilon^{ab} B_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu \right.$$

$$\left. + \alpha' R \Phi(X) \right)$$ Find $T_a{}^a = -\frac{1}{2\alpha'} \beta^G_{\mu\nu} g^{ab} \partial_a X^\mu \partial_b X^\nu$

$$- \frac{i}{2\alpha'} \beta^B_{\mu\nu} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \beta^\Phi R$$

With $\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + O(\alpha'^2)$

$$\beta^B_{\mu\nu} = -\frac{\alpha'}{2} \nabla^\rho H_{\rho\mu\nu} + \alpha' \nabla^\rho \Phi H_{\rho\mu\nu} + O(\alpha'^2)$$

$$\beta^\Phi = \frac{D-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\rho \Phi \nabla^\rho \Phi$$

$$- \frac{\alpha'}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} + O(\alpha'^2)$$

some sol'ns: ① $D = 26$, $G_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = 0$, $\Phi = \Phi_0$

For $D < 26$, another sol'n is

② $D < 26$, $G_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = 0$, $\Phi = V_\mu X^\mu$

with $V_\mu V^\mu = \frac{26 - D}{6\alpha'}$. "linear dilaton vacuum"

Not translationally inv't in V_μ direction. This is the Liouville field in $h_{ab} = e^\phi \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ which gets a kinetic term for $c_{Total} \neq 0$.

Can also consider sol'ns with $D = 26$, $\Phi = \Phi$ const, $B_{\mu\nu} = 0$, and more general metric $G_{\mu\nu}$ with $R_{\mu\nu} = 0$

$\rightarrow$ sol'n of vacuum satisfying Einstein's eqns.

To higher order in α' get corrections to $\beta^G_{\mu\nu}$ from higher powers of curvature & α' $\longrightarrow$ corrections to Einstein's eqns from string theory, e.g. $\beta^G_{\mu\nu} = -\alpha'(R_{\mu\nu} + \frac{\alpha'}{2} R_{\mu\rho\sigma\tau} R_{\nu}{}^{\rho\sigma\tau})$

The $T_a{}^a = 0$ vanishing β function eqns of worldsheet give the spacetime equations of motion. They follow from varying a spacetime effective action:

$$S_{spacetime} = \frac{1}{2\kappa_0^2} \int d^D x \sqrt{-\det G}\, e^{-2\Phi} \Big(-\frac{2(D-26)}{3\alpha'} + \mathcal{R}_{(spacetime)} - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\, \partial_\mu \Phi\, \partial^\mu \Phi + O(\alpha') \Big)$$

Term $\frac{1}{2\kappa_0^2} \int d^D x \sqrt{-\det G}\, e^{-2\Phi} \mathcal{R}_{\text{spacetime}}$

is the D dim'l Einstein Hilbert action,

with Brans Dicke coupling to dilaton scalar

Φ.

Classical string eqns of motion from

$\beta = 0$ for massless fields:

$\Phi = \Phi_{cl} + \phi$ with $\langle \phi \rangle = 0$

in string thy $\longleftrightarrow$ $\langle V \rangle = 0$

$\uparrow$ vertex op.

On sphere consequence of conformal inv.

$z \to \lambda z$
$\bar{z} \to \bar{\lambda} \bar{z}$ $\longrightarrow$ $V \to |\lambda|^{-2} V$ since $(h, \bar{h}) = (1,1)$

so $\langle V \rangle \longrightarrow |\lambda|^{-2} \langle V \rangle$ symm only

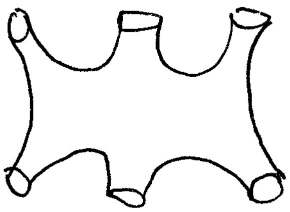
if $\langle V \rangle = 0$

Tadpoles $\langle V \rangle$ for massless fields must

vanish. (If massive $\sim \frac{g}{M^2}$ ← from prop. just shift

vacuum.)

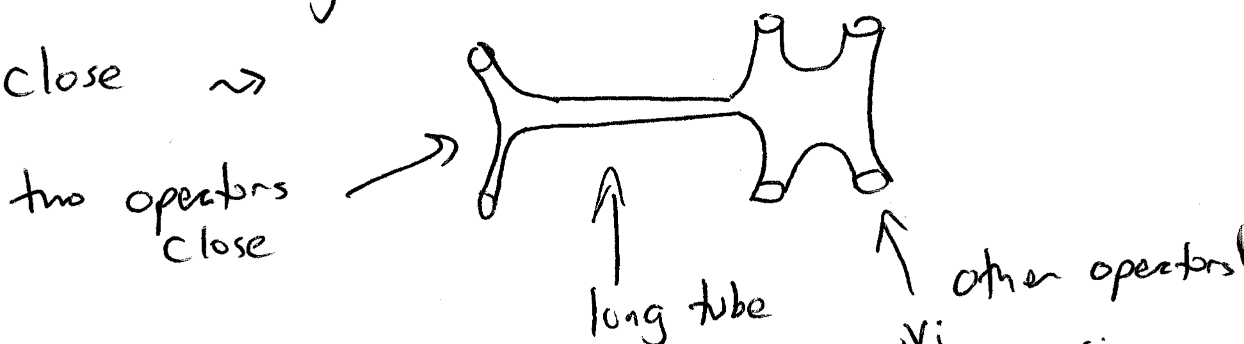
Consider string scattering amplitudes



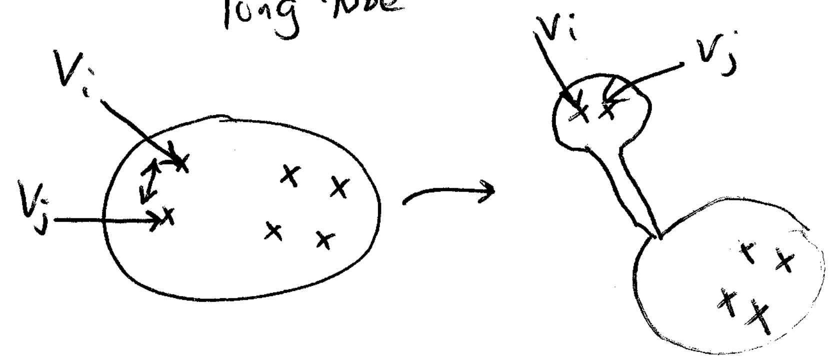
tree level $\rightarrow$ $\left\langle \prod_{i=1}^{n} \int d^2 z_i \; V_i(z_i) \right\rangle$

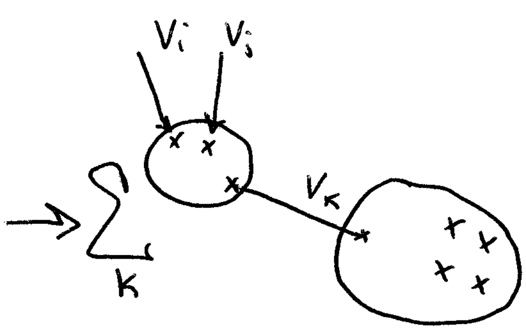
n particle scattering amp.

(will have to modify slightly to account for Mobius symmetry, discuss this shortly). Consider regions of integration where two z_i are very close $\rightsquigarrow$



Can picture as





This is the statement of the operator product expansion:

replace $V_i(z_i)\, V_j(z_j) \longrightarrow \sum_k \dfrac{C_{ij}{}^{k} \quad \mathcal{O}_k(z_j)}{(z_i - z_j)^{h_i + h_j - h_k} \, (\bar{z}_i - \bar{z}_j)^{\bar{h}_i + \bar{h}_j - \bar{h}_k}}$

$\mathcal{O}_k$ = basis of operators

In this limit replace e.g.

$$\int d^2 z_i \; V_i(z_i)\, V_j(z_j) \longrightarrow \int d^2 y \sum_K \frac{C_{ij}{}^K \; \mathcal{O}_K(z_j)}{(y)^{h_i+h_j-h_K} (\bar{y})^{\bar{h}_i+\bar{h}_j-\bar{h}_K}}$$

$y \equiv z_i - z_j \to 0$

Get singularity when $h_i + h_j - h_K = \bar{h}_i + \bar{h}_j - \bar{h}_K = 1$

from $$\int \frac{d^2 y}{|y|^{2(h_i+h_j-h_K)}} = 2\pi \int \frac{r\,dr}{r^{2(h_i+h_j-h_K)}}$$

taking $h_i = \bar{h}_i$

$\sim \ln r \big|_{r\to 0}$ for $h_i + h_j - h_K = 1$ $\rightsquigarrow$ $\dfrac{1}{h_i + h_j - h_K - 1}$

Since $h_i = h_j = 1$ for physical Vertex operators

we get $\sim \dfrac{1}{h_K - 1}$ poles for internal states

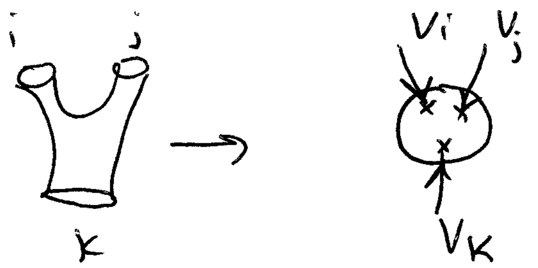
= physical on shell string states ✓

Writing $h_K = \dfrac{p_K^2}{2} + N =$ ~~[illegible]~~

$= \frac{1}{2}(p_K^2 + m_i^2) + 1$ singularity $\sim \dfrac{1}{p_K^2 + m_K^2}$ as usual,

propagator for internal state

Consider the 3 point interaction vertex



By Mobius transform

we can map eg

$z_i \to 0 \qquad z_j \to 1 \qquad z_k \to \infty$

get $\langle V_k | V_j(1) | V_i \rangle \sim C_{ij}{}^k = \text{const}$

OPE coeffs $C_{ij}{}^k \to$ basic interaction vertices.

On the otherhand, if we tried to compute $\left\langle \int d^2 z_i \int d^2 z_j \int d^2 z_k \, V_i(z_i) V_j(z_j) V_k(z_k) \right\rangle$

we would get $\infty \sim$ volume of group of Mobius transformations. For primary V_i, V_j, V_k with $h_i = \bar{h}_i = h_j = \bar{h}_j = h_k = \bar{h}_k = 1$,

$\langle V_i(z_i) V_j(z_j) V_k(z_k) \rangle = C_{ijk} \big/ |z_i - z_j|^2 |z_i - z_k|^2 |z_j - z_k|^2$

So want to cancel $\iiint \frac{d^2z_i\, d^2z_j\, d^2z_k}{|z_i-z_j|^2\,|z_i-z_k|^2\,|z_j-z_k|^2} \to 1$

We can understand this in terms of the infinitesimal Mobius transformations $z \to \lambda_{-1} + \lambda_0 z + \lambda_1 z^2$

$\lambda_{-1}, \lambda_0, \lambda_1$ complex parameters.

$$d^2z_i\, d^2z_j\, d^2z_k = d^2\lambda_{-1}\, d^2\lambda_0\, d^2\lambda_1 \, \det\left(\frac{\partial z}{\partial \lambda}\right) \det\left(\frac{\partial \bar{z}}{\partial \bar{\lambda}}\right)$$

$$\frac{\partial z}{\partial \lambda} = \begin{vmatrix} \frac{\partial z_1}{\partial \lambda_{-1}} & \frac{\partial z_2}{\partial \lambda_1} & \frac{\partial z_3}{\partial \lambda_1} \\ \frac{\partial z_1}{\partial \lambda_0} & \frac{\partial z_2}{\partial \lambda_0} & \frac{\partial z_3}{\partial \lambda_0} \\ \frac{\partial z_1}{\partial \lambda_1} & \frac{\partial z_2}{\partial \lambda_1} & \frac{\partial z_3}{\partial \lambda_1} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{vmatrix}$$

(oops $1,2,3 = i\,j\,k$)

$$= (z_i - z_j)(z_i - z_k)(z_j - z_k)$$

So $\quad d^2z_i\, d^2z_j\, d^2z_k = |z_i-z_j|^2\,|z_i-z_k|^2\,|z_j-z_k|^2\, d^2\lambda_1\, d^2\lambda_0\, d^2\lambda_1$

to avoid getting ∞ we just drop the integrals over $d^2\lambda_{-1}\; d^2\lambda_0\; d^2\lambda_1$

which give the ∞ volume of $SL(2,\mathbb{C})$ Mobius transfs.

More generally for n point amplitudes replace $\langle \prod_{i=1}^{n} \int d^2 z_i \, V_i(z_i) \rangle$

with

$$|z_1-z_2|^2 \, |z_1-z_3|^2 \, |z_2-z_3|^2 \left(\prod_{i=4}^{n} \int d^2 z_i \right) \langle \prod_i V_i(z_i) \rangle$$

dropped 3 integrals & included determinant factor.

Likewise for open strings we replace

$\langle \prod_{i=1}^{n} \int_{\partial\Sigma} dz_i \, V_i(z_i) \rangle$ with

$$(z_1-z_2)(z_2-z_3)(z_3-z_1) \left(\prod_{i=1}^{4} \int dz_i \right) \langle \prod_j V_j(z_j) \rangle$$

The $|z_1-z_2|^2 |z_1-z_3|^2 |z_2-z_3|^2$ or

$(z_1-z_2)(z_2-z_3)(z_3-z_1)$ factors are examples of Faddeev Popov determinants.

Example: tachyon scattering $V_i(z_i,\bar{z}_i) = e^{ip_i\cdot X(z_i,\bar{z}_i)}$

All $\frac{\alpha'}{2}\frac{p_i^2}{2} = 1$.

$$\left\langle \prod_{i=1}^{n} V_i(z_i,\bar{z}_i)\right\rangle = \delta^D\left(\sum_{i=1}^{n} p_i^\mu\right)\prod_{i<j=1}^{n}(z_i-z_j)^{p_i\cdot p_j\alpha'/2}$$

(+ complex conj)

so Amplitude $A(p_i) = g_s^n\, \delta^D\left(\sum_{i=1}^{n} p_i^\mu\right)$

$$|(z_1-z_2)(z_2-z_3)(z_3-z_1)|^2 \int d^2z_4 \cdots d^2z_n \prod_{i<j=1}^{n}|z_i-z_j|^{p_i\cdot p_j\alpha'}$$

Usually take $z_1\to 0$ $z_2\to 1$ $z_3\to\infty$

(all z_1,z_2,z_3 dep drops out anyway).

Note for large z_3, get terms

$$|z_3|^4 \prod_{\substack{i\neq 3\\ i=1}}^{n} |z_3|^{p_i\cdot p_3\alpha'} = |z_3|^{(4-p_3^2\alpha')}$$

(using $\sum_{i\neq 3} p_i = -p_3$ momentum conservation)

$= 1$ ← (since $p_3^2 = 4/\alpha'$)

So z_3 dep properly drops out using momentum conservation for on shell amplitude $p_3^2 = 4/\alpha'$!

E.g. for $n=4$ point functions w/ $(z_1, z_2, z_3) \to (0, 1, \infty)$

$$A = g_s^4 \, \delta^{(D)}\left(\sum p_i\right) \int d^2 z_4 \, |z_4|^{\alpha' p_1 \cdot p_4} \, |1 - z_4|^{\alpha' p_2 \cdot p_4}$$

$s = -(p_1+p_2)^2 = 2m^2 - 2p_1 \cdot p_2$ $\qquad m^2 = -4/\alpha'$

$t = -(p_1+p_4)^2 = 2m^2 - 2p_1 \cdot p_4$

$u = -(p_2+p_4)^2 = 2m^2 - 2p_2 \cdot p_4$

$$A(s,t,u) = g_s^4 \, \delta^D\left(\sum p_i\right) \underbrace{\int d^4 z_4 \, |z_4|^{-(\alpha' t + 8)/2} \, |1 - z_4|^{-(\alpha' u + 8)/2}}_{*}$$

$$* = 2\pi \frac{\Gamma\left(-\frac{1}{2}\alpha(s)\right) \Gamma\left(-\frac{1}{2}\alpha(t)\right) \Gamma\left(-\frac{1}{2}\alpha(u)\right)}{\Gamma\left(-\frac{1}{2}(\alpha(s) + \alpha(t))\right) \Gamma\left(-\frac{1}{2}\alpha(s) - \frac{1}{2}\alpha(u)\right) \Gamma\left(-\frac{1}{2}\alpha(t) - \frac{1}{2}\alpha(u)\right)}$$

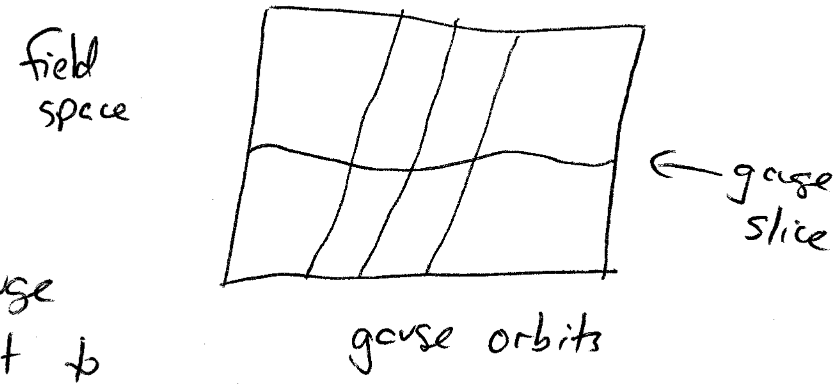
$s \leftrightarrow t \leftrightarrow u$ symmetry $\qquad \alpha(s) \equiv \frac{\alpha'}{2} s + 2$ etc.

Poles at $\alpha(s)$, $\alpha(t)$, or $\alpha(u) = 0, 2, \ldots$

on shell intermediate states at $s, t,$ or u

$= +4/\alpha' (N-1) \leftarrow$ exactly on shell mass cond!

Faddeev Popov:



○ To avoid overcounting must divide out by gauge orbit symmetry, restrict to a gauge slice. Get Jacobian determinant. Here gauge orbit from $(\sigma, \tau) \to (\sigma'(\sigma,\tau), \tau'(\sigma,\tau))$ reparam symm. Inf version acts on worldsheet metric $h_{ab} \to h_{ab} + \nabla_a V_b + \nabla_b V_a$. Fix by setting $h_{ab} = e^{\phi} \hat{h}_{ab}$ ← fixed ref metric eg η_{ab}

○ Variations about slice with $h_{zz} = h_{\bar{z}\bar{z}} = 0$

$$\delta h_{zz} = \nabla_z V_z \qquad \delta h_{\bar{z}\bar{z}} = \nabla_{\bar{z}} V_{\bar{z}}$$

$$Dh_{zz}\, Dh_{\bar{z}\bar{z}} = \det \nabla_z \det \nabla_{\bar{z}}\, DV_z\, DV_{\bar{z}}$$

replace $\displaystyle\int \frac{Dh_z\, Dh_{\bar{z}\bar{z}}}{\text{Volume gauge orbit}} = \int \cancelto{1}{\frac{DV_z\, DV_{\bar{z}}}{\text{Vol(orbit)}}} \det \nabla_z \det \nabla_{\bar{z}}$

get factors of $\det \nabla_z \det \nabla_{\bar{z}} = \int [db\,dc]\, e^{-S_{ghost}}$

○ $$S_{ghost} = \frac{1}{2\pi} \int d^2 z \left(\underset{(h,\bar{h}) = (2,0)}{b_{zz}}\, \bar{\partial}_{\bar{z}} \underset{(h,\bar{h}) = (-1,0)}{c^z} + \underset{(0,2)}{\bar{b}_{\bar{z}\bar{z}}}\, \partial_z \underset{(0,-1)}{\bar{c}^{\bar{z}}} \right)$$

Write $b(z) = \sum_{n=-\infty}^{\infty} \frac{b_n}{z^{n+2}}$ $\quad c(z) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n-1}}$

$[db\,dc] = \prod_n db_n \prod_m dc_n$

$\{b_n, b_m\} = 0$

$\{c_n, c_m\} = 0$

$\left(\text{from } b(z)c(w) \sim \frac{1}{z-w}\right) \longrightarrow \{b_n, c_m\} = \delta_{n+m,0}$

Recall anticommuting coordinates θ, $\theta^2 = 0$

$f(\theta) = f(0) + f'(0)\,\theta$ Simple Taylor exp.

$\int d\theta\; 1 = 0 \qquad \int d\theta\; \theta = 1$

so $\int db_n\, dc_n\; e^{-\lambda_n b_n c_n} = \lambda_n$

$$\int \prod_n db_n \prod_m dc_m\; e^{-M_{nm} b_n c_m} = \det M$$

Why $\int [db\,dc]\, e^{-S_{ghost}} \longrightarrow \det \nabla_z \det \nabla_{\bar{z}}$