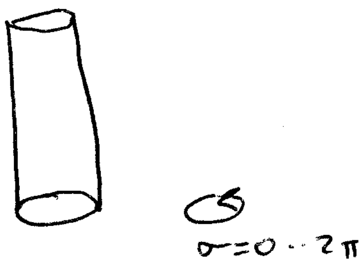
For $f = z + \varepsilon(z)$ with ε inf. $\quad T \to T + \delta_\varepsilon T$

with $\delta_\varepsilon T(z) = \left(2(\partial_z \varepsilon) + \varepsilon \partial_z\right) T + \frac{c}{12} \partial^3 \varepsilon(z)$

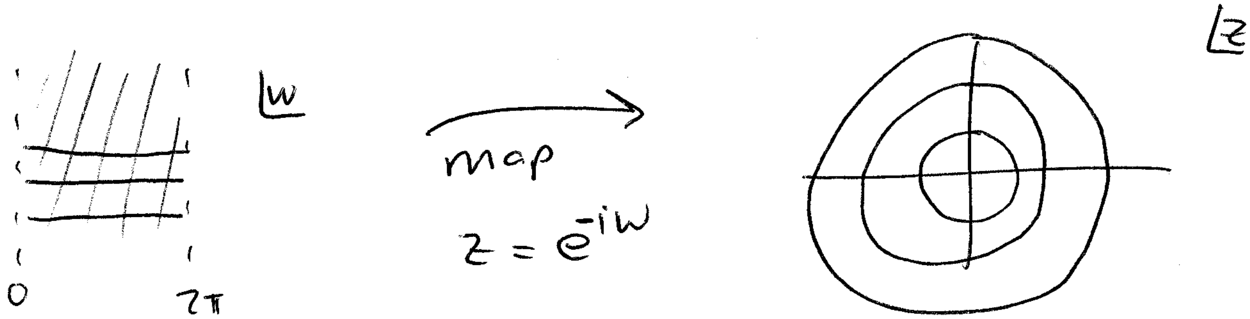
c is the "central charge" or "conformal anomaly" of theory = real number specific to theory.

String moving in time $\to$ cylinder



$\uparrow \tau = -\infty \cdots \infty$ $\qquad$ let $w = \sigma + i\tau$

$\sim w + 2\pi$



equal worldsheet τ time $\longrightarrow$ equal radius in z plane

radial quantization.

$$T_{ww} = \left(\frac{\partial z}{\partial w}\right)^2 T_{zz} + \frac{c}{12}\{z, w\}$$

$$= -z^2 T_{zz} + \left(\frac{c}{12}\right)\left(\frac{+1}{2}\right)$$

$$T_{ww} = -\sum_{m=\infty}^{\infty} T_m \left(e^{-iw}\right)^m \qquad T_{zz} = \sum_{m=-\infty}^{\infty} \frac{L_n}{z^{n+2}}$$

Fourier modes

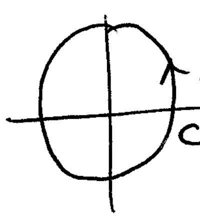
$$T_m = L_m - \delta_{m,0}\frac{c}{24}, \qquad \overline{T}_m = \overline{L}_m - \delta_{m,0}\frac{\overline{c}}{24}$$

Hamiltonian: $H = \int_0^{2\pi} \frac{d\sigma}{2\pi} T_{\tau\tau} = T_0 + \overline{T}_0$

$$\boxed{H = L_0 + \overline{L}_0 - \frac{(c+\overline{c})}{24}.}$$

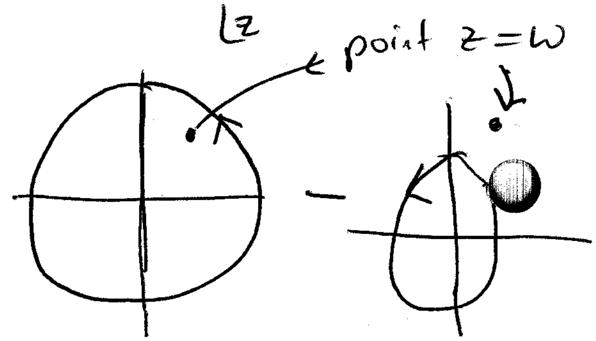
Casimir Energy
$E = -\frac{(c+\overline{c})}{24} \Rightarrow -\frac{\pi(c+\overline{c})}{12\ell}$
(ℓ = length)

Conserved charges $\quad Q = \oint_C \frac{dz}{2\pi i} j, \quad \overline{Q} = \text{c.c.}$



contour ~ "equal time slice"

Commutator $[Q, \mathcal{O}(w)] =$

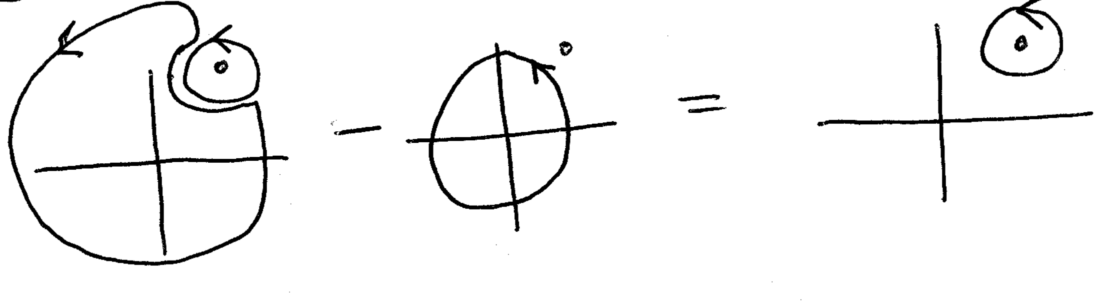


radial ordering $Q\,\mathcal{O}(w) \Longrightarrow$ Q contour is outside $\mathcal{O}$ insertion point w, ie $|z| > |w|$

like time ordered prod.

$\mathcal{O}(w)\,Q \Longrightarrow |w| > |z|.$

Commutator is difference =



$\Rightarrow \quad [Q, \mathcal{O}(w,\bar{w})] = \oint_C \frac{dz}{2\pi i}\, j(z)\, \mathcal{O}(w,\bar{w})$ where

- C surrounds point z. Take $z-w$ small & use

Operator Product expansion :

$$\lim_{z\to w} \mathcal{O}_i(z,\bar{z})\, \mathcal{O}_j(w,\bar{w}) = \sum_K \frac{C_{ij}^{\ K}\ \mathcal{O}_K(w,\bar{w})}{(z-w)^{h_i+h_j-h_k}(\bar{z}-\bar{w})^{\bar{h}_i+\bar{h}_j-\bar{h}_k}}$$

With $C_{ij}^{\ K}$ = constants & $\mathcal{O}_i$ operators with dim h_i.

For Φ primary of dims h & $\bar{h}$

- $\delta_{\varepsilon\bar{\varepsilon}} \Phi = (h\partial\varepsilon + \bar{h}\partial\bar{\varepsilon} + \varepsilon\partial + \bar{\varepsilon}\bar{\partial})\Phi$

$$= \oint \frac{dz}{2\pi i}\, \varepsilon(z) T(z) \Phi(w,\bar{w}) + \oint \frac{d\bar{z}}{2\pi i}\, \bar{\varepsilon}(\bar{z}) \bar{T}(\bar{z}) \Phi$$

$$\Rightarrow \quad T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{\partial_w \Phi(w)}{(z-w)}$$

+ nonsing.

* Show this & $L_n = \oint \frac{dz}{2\pi i}\, z^{n+1}\, T(z) \Rightarrow$

- $[L_n, \Phi(w)] = \left(h(n+1)w^n + w^{n+1}\frac{\partial}{\partial w}\right)\Phi(w,\bar{w})$

$L_{-1} \to \frac{\partial}{\partial w}$ translations $\quad L_0 = h + w\frac{\partial}{\partial w}$ dilatations

$L_1 = 2hw + w^2\partial_w$ special conf'l

$$\delta_\varepsilon T(w) = (2\partial\varepsilon + \varepsilon\partial)T + \frac{c}{12}\partial^3\varepsilon$$

$$= \oint \frac{dz}{2\pi i}\, T(z)\,\varepsilon(z)\, T(w)$$

z surrounds w.

$$\Rightarrow \quad T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

$$\Rightarrow \quad \langle T(z)T(w)\rangle = \frac{c/2}{(z-w)^4}$$

"central term" ~ operator 1 in O.P.E.

Can show $\bar\partial\left(\frac{1}{z}\right) = \partial\left(\frac{1}{\bar z}\right) = \pi\,\delta^2(z,\bar z)$

e.g. $\int d^2z\, \bar\partial v = \oint \frac{dz}{2i}\, v$ for $v = \frac{1}{z}$

($2i$ since $z = \sigma^1 + i\sigma^2$ so $dz\wedge d\bar z = 2i\, d\sigma_1 \wedge d\sigma_2$ etc.)

so $\partial_{\bar z}\langle T(z)T(w)\rangle = -\partial_{\bar z}\left(\frac{c}{12}\right)\partial_z^3 \frac{1}{(z-w)}$

$$= -\frac{c}{12}\,\partial_z^3\, \pi\,\delta^2(z-w)$$

$$= -\partial_z\langle T_{z\bar z}T(w)\rangle$$

$\Rightarrow$ $\langle T_{z\bar z}(z)T(w)\rangle =$

$$= \frac{c}{12}\,\pi\,\partial_z^2\,\delta^2(z-w,\bar z-\bar w)$$

Active / passive version of

$$\delta_\varepsilon T = \left(2(\partial\varepsilon) + \varepsilon\partial\right)T + \frac{c}{12}\partial^3\varepsilon$$

under $h_{ab} \to h_{ab} + \delta h_{ab}$ transf of 2d worldsheet metric

$$\partial_{\bar z}\langle T_{zz}\rangle = \frac{c}{48}\partial_z^3\,\delta h^{zz}$$

Use $\partial_{\bar z} T_{zz} + \partial_z T_{\bar z z} = 0$

$$\hookrightarrow \quad \partial_z\langle T_{z\bar z}\rangle = -\frac{c}{48}\partial_z^3\, g^{zz}$$

$$\hookrightarrow \quad \langle T_{z\bar z}\rangle = -\frac{c}{12} R$$

$\uparrow$ scalar curv. of 2d worldsheet

$c =$ conformal anomaly $\Rightarrow$ lack of inv.

under $h_{ab} \to e^{\phi} h_{ab}$. $\quad \sqrt{\det h}\,R \to \sqrt{\det h}\,(R - \partial^a\partial_a\phi)$

Action $S_{eff} \to S_{eff} + \left(\frac{c}{24}\right)\left(\frac{1}{4\pi}\right)\int d^2\sigma\sqrt{-\det h}\, h^{ab}\partial_a\phi\,\partial_b\phi$

i.e. for $c \neq 0$ the Liouville field which we eliminated classically by a Weyl rescaling comes back to life, with its own kinetic term for $c \neq 0$!

Note general 2d CFT can have $c_L (\equiv c) \neq c_R (\equiv \bar{c})$. But such theories can not be consistently be coupled to gravity: $c_L - c_R \neq 0 \Rightarrow$ there is a gravitational anomaly.

For $c_L = c_R \neq 0$ can couple to gravity but Liouville field becomes dynamical. Will see later that turning on gravity introduces Faddeev Popov ghosts which also contribute to c. Liouville field decouples iff $c_{ghosts} + c_{matter} = 0$.

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots$$

$$\Rightarrow \underbrace{[L_n, L_m]}_{\parallel} = (n-m)\,L_{n+m} + \frac{c}{12}(n^3-n)\,\delta_{n+m,0}$$

use $\displaystyle \oint \frac{dz}{2\pi i}\, z^{n+1} \oint \frac{dw}{2\pi i}\, w^{m+1}\; T(z)T(w)$

$\uparrow$ z around w

This is the "Virasoro alg."

note L_0, L_1, L_{-1} form closed subalg &

central term drops out for $n = 0, \pm 1$.

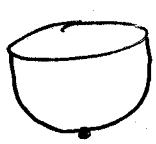
$$[L_0, L_m] = -m\,L_m$$

$\Rightarrow$ L_m lowers L_0 eigenvalue by m

$L_{m>0}$ = annihilation op $\quad L_{m<0}$ = creation ops

Operators $\longrightarrow$ states mapping

1 identity op $\longrightarrow$ $|0\rangle$ = in vacuum state



in state inserted down cylinder $\tau \to -\infty$

General in state $|\phi\rangle = \lim_{|z|\to 0} \phi(z,\bar{z})\,|0\rangle$

i.e.  $\leftarrow |\phi\rangle$ state here (fixed τ time circle)

$\leftarrow \phi$ inserted here

Likewise out states $\langle 0| =$ 

and $\langle \phi| = \lim_{w\to\infty} \langle 0|\, \tilde{\phi}(w,\bar{w})$

With $w \Rightarrow 1/z$: $\phi \to \tilde{\phi}$  $\leftarrow \tilde{\phi}$ $\langle\phi|$

adjoint via $z \to 1/z$

$$(|\phi\rangle)^{+} = \langle\phi| \qquad L_m^{+} = L_{-m}$$
$$\bar{L}_m^{+} = \bar{L}_{-m}$$

For ϕ primary op of dim. h

$$L_0|\phi\rangle = h|\phi\rangle$$
$$L_{n>0}|\phi\rangle = 0$$

$L_{n>0}$ annihilate $L_{n<0}$ creation ops

e.g. $\prod_i L_{-n_i}\,|\phi\rangle \leftarrow$ "descendents"

has $L_0 = h + \sum_i n_i$ but not primary since $L_{n>0}$ don't all annihilate

Can expand $\phi(z) = \sum_{n \in \mathbb{Z} - h} \frac{\phi_n}{z^{n+h}}$

$$\phi_n = \oint \frac{dz}{2\pi i} z^{n+h-1} \phi(z)$$

$|\phi\rangle = \phi_{-h} |0\rangle \qquad \phi_{n \geq -h+1} |0\rangle = 0$

can build general state $\prod_i \phi_{-n_i - h} |0\rangle \quad n_i \geq 0$

For primary $\phi \qquad [L_n, \phi_m] = (n(h-1) - m) \phi_{n+m}$

so $[L_0, \phi_{-m}] = m \phi_{-m}$ raises L_0 eigenvalue by m

Primary states are orthogonal to descendents.

$|\chi\rangle$ descendent $\quad |\chi\rangle = L_{-n} |\psi\rangle$ for some $n > 0$

$|\psi\rangle$. For primary $|\phi\rangle$,

$$\langle \phi | \chi \rangle = \langle \phi | L_{-n} | \psi \rangle = 0$$

since $\langle \phi | L_{-n} = 0$ for $n > 0$

Can also show some unitarity rel'ns:

$$\langle\phi| L_{-n}^{+} L_n |\phi\rangle = \| L_{-n} |\phi\rangle \|^2$$

$$= \langle\phi| [L_n, L_{-n}] |\phi\rangle$$

$$= 2n \langle\phi| L_0 |\phi\rangle + \frac{c}{12}(n^3-n) \langle\phi|\phi\rangle$$

$$= \left(2nh + \frac{c}{12}(n^3-n)\right) \| |\phi\rangle \|^2$$

In unitary thy all $\| |\chi\rangle \|^2 \geq 0$

so need $2nh + \frac{c}{12}(n^3-n) \geq 0$

$n=1 \Rightarrow h \geq 0$ with $h=0$ iff $L_{-1}|\phi\rangle = 0$

$n>1 \Rightarrow c \geq 0$

Will later encounter Faddeev Popov ghosts which have $c = -26$, $\therefore$ not unitary.

Vacuum state $|0\rangle$ has $L_{n\geq -1}|0\rangle = 0$

$\bar{L}_{n\geq -1}|0\rangle = 0$

also $\langle 0| L_{m\leq 1} = 0$ $\quad$ $\langle 0| \bar{L}_{m\leq 1} = 0$

Subalg: L_{-1}, L_0, L_1 & $\bar{L}_{-1}, \bar{L}_0, \bar{L}_{+1}$

These annihilate both $|0\rangle$ & $\langle 0|$. Symm. of theory. These are the infinitesimal generators of $SL(2,\mathbb{R}) \times \overline{SL(2,\mathbb{R})} \simeq SL(2,\mathbb{C})$

recall $L_n: \quad z \longrightarrow z + \varepsilon z^{n+1}$

so $\quad L_{-1}: \quad z \to z+\varepsilon \quad$ translation

$L_0: \quad z \to z + \varepsilon z \quad$ scale transf.

$L_1: \quad z \to z + \varepsilon z^2 \quad$ "special conf. transf."

Exponentiate, these generate Mobius transformations

$$z \longrightarrow \frac{az+b}{cz+d}$$

with a, b, c, d real
$(ad-bc) = 1$

i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ real entries with unit determinant

Likewise $\bar{z} \to \dfrac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}$ with $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \in SL(2,\mathbb{R})$

Group both together as $z \to \dfrac{az+b}{cz+d}$ w/ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$

complex

* Show that $z \to \dfrac{az+b}{cz+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$

symm transformations forms a <u>group</u> with

group multiplication obtained by composition

of $z \to \dfrac{az+b}{cz+d}$, is same as matrix

multiplication of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$g_1: \ z \to \dfrac{a_1 z+b_1}{c_1 z+d_1}$ $\qquad g_2: \ z \to \dfrac{a_2 z+b_2}{c_2 z+d_2}$

$g_1 g_2$: first do g_1 & then g_2 transforms

$z \to \dfrac{a_3 z+b_3}{c_3 z+d_3}$ with $\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

Show that this is also an element of $SL(2,\mathbb{R})$

and that every element has an inverse

g^{-1} with $g g^{-1} = g^{-1} g = I$ identity elem.

Recall primary ops under ~~conformal~~ $z \to f(z)$

$$\Phi(z,\bar{z}) \longrightarrow \left(\frac{\partial f}{\partial z}\right)^{h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi\left(f(z), \bar{f}(\bar{z})\right)$$

Their correlatn fns $\therefore$ satisfy

$$\left\langle \prod_{i=1}^{n} \Phi_i(z_i,\bar{z}_i) \right\rangle \longrightarrow \prod_{i=1}^{n} \left(\frac{\partial f}{\partial z}\right)^{h_i} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}_i} \Bigg|_{z=z_i} \cdot$$

$$\cdot \left\langle \prod_{i=1}^{n} \Phi_i\left(f(z_i), \bar{f}(\bar{z}_i)\right) \right\rangle$$

For 2 point functions this $\Rightarrow$

$$\langle \Phi_i(z,\bar{z}) \Phi_j(w,\bar{w}) \rangle = \frac{C_{ij} \quad \delta_{h_i,h_j}}{(z-w)^{2h_i} (\bar{z}-\bar{w})^{2\bar{h}_j}} \delta_{\bar{h}_i,\bar{h}_j}$$

with C_{ij} = constants

For 3 point functions $\langle \Phi_i(z_1,\bar{z}_1) \Phi_j(z_2,\bar{z}_2) \Phi_k(z_3,\bar{z}_3) \rangle$

$$= C_{ijk} \frac{1}{(z_1-z_2)^{h_i+h_j-h_k} (z_1-z_3)^{h_i+h_k-h_j} (z_2-z_3)^{h_j+h_k-h_i}}$$

$\cdot$ (complex conj) again C_{ijk} = constants.

4 point functions

$$\left\langle \prod_{i=1}^{4} \Phi_i(z_i, \bar{z}_i) \right\rangle = f(x, \bar{x})$$

$$\prod_{i<j} z_{ij}^{-(h_i+h_j)+h/3} \quad \prod_{i<j} \bar{z}_{ij}^{-(\bar{h}_i+\bar{h}_j)+\bar{h}/3}$$

$$z_{ij} \equiv z_i - z_j \qquad h \equiv \sum_{i=1}^{4} h_i$$

$$x \equiv z_{12} z_{34} / z_{13} z_{24}$$ $f(x, \bar{x})$: arbitrary function

cross ratio

How to understand why $n \leq 3$ point functions are more constrained: Can use Mobius transformations $z \longrightarrow \frac{az+b}{cz+d}$ to map any 3 pts to any 3 other points, e.g. any 3 points to $0, 1, \infty$

$z_1 \longrightarrow \infty \qquad z_2 \longrightarrow 1 \qquad z_4 \longrightarrow 0$

(radial ordering). 4th point is (z_3 here) unconstrained, hence arb fnctn $f(x, \bar{x})$

$z_3 \to x$ invt. under Mobius transformations.