World line of particles $(- + + + \cdots +)$

$$S_{pp} = -m\int d\tau \left(-\dot{X}^\mu \dot{X}_\mu\right)^{1/2} \qquad \left(\dot{} = \frac{\partial}{\partial\tau}\right)$$

$\left(= -m\int ds \leftarrow \text{length of path}\right)$

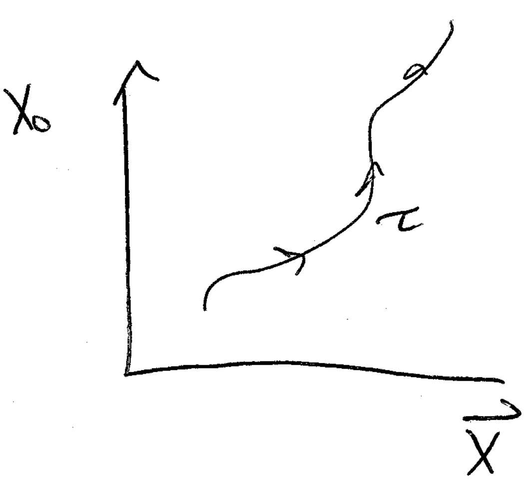
* show in non rel limit $S \approx \int dt\,(T - V)$

with $T = \frac{1}{2} m \left(\frac{d\vec{x}}{dt}\right)^2, \quad V = m$

- S_{pp} EOM: $\dot{u}^\mu = 0 \qquad u^\mu \equiv \dot{X}^\mu \left(-\dot{X}^\nu \dot{X}_\nu\right)^{-1/2}$ – velocity
- S_{pp} invt under $\tau \longrightarrow \tau' = \tau'(\tau)$ general coord transf.

$$S = \frac{1}{2}\int d\tau \left(\eta^{-1} \dot{X}^\mu \dot{X}_\mu - \eta m^2\right)$$

$\eta \sim \sqrt{h}$ metric of τ worldline



$$\eta'(\tau')\,d\tau' = \eta(\tau)\,d\tau$$

η EOM: $\eta^{-2}\dot{X}^\mu \dot{X}_\mu = -m^2$

For $m \neq 0$ solve for η

$$\eta^2 = -\dot{X}^\mu \dot{X}_\mu / m^2$$

Plug this η back in $S \Rightarrow$ $S \to S_{pp} = -m \int d\tau \sqrt{-\dot{X}_\mu \dot{X}^\mu}$

- In quantum mech. $S = \frac{1}{2}\int d\tau \left(\eta^{-1} \dot{X}^\mu \dot{X}_\mu - \eta m^2\right)$ is easier to handle than S_{pp} since quadratic in $\dot{X}^\mu$, path integrals are gaussian integrals easiest case. Eliminate η at the end of the day.

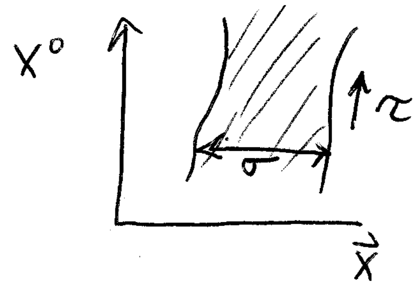
Generalize to strings / membranes / etc. ?

- $S_{NG} = -T \int d^n\sigma \sqrt{-\det \tilde{h}_{ab}}$ ($\sim$ area or volume of world sheet or volume)

$\tilde{h}_{ab} = g_{\mu\nu}\, \partial_a X^\mu\, \partial_b X^\nu$ induced metric

σ^a World sheet / volume coords, $a = 1 \ldots n$

$X^\mu = X^\mu(\sigma^a)$ es



- S_{NG} is the n dim'l area ~~[illegible]~~

T is the tension $\sim (\text{mass})^n$

Try the same trick as before to get a quadratic action, i.e. try replacing S_{NG} with

$$S_{\text{polyakov}} = -\frac{T}{2}\int d^n\sigma \underset{(\equiv \det h)}{\sqrt{-h}}\; h^{\alpha\beta} g_{\mu\nu}\, \partial_\alpha X^\mu \partial_\beta X^\nu$$

where now $h^{\alpha\beta}$ is an indep field whose EOM we want to give a constraint, which, when imposed, give $S_{\text{polyakov}} \to S_{NG}$.

Works <u>only</u> for $n=2$ <u>Strings</u> are special!

in S_{polyakov}, $h_{\alpha\beta} \to \sim n$ dimensional gravity theory. $h_{\alpha\beta}$ has $\frac{n(n+1)}{2}$ compts minus n reparametrizations $\to \frac{n(n-1)}{2}$ compts left.

Can eliminate for $n=1$. For $n \geq 3$ $h_{\alpha\beta}$ has nontrivial dynamics, can't eliminate. $n=2$ is special case. $h_{\alpha\beta} \to 3$ compts. Use reparam to take $h_{\alpha\beta} = e^\phi \eta_{\alpha\beta}$, ϕ a'priori can't be eliminated. But ϕ drops out in $\sqrt{\det h}\, h^{\alpha\beta}$, special to $n=2$!

For $n=2$, strings, $T \sim m^2 \sim \ell^2$ tension. $T \equiv 1/2\pi\alpha'$

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\det h_{\alpha\beta}}\, h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}$$

$h_{\alpha\beta}$ EOM $\Rightarrow$ $h_{\alpha\beta} = g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \equiv \tilde{h}_{\alpha\beta}$

$$S_P \to -\frac{1}{4\pi} \int d^2\sigma \sqrt{-\det \tilde{h}_{\alpha\beta}}\, h^{\alpha\beta} h_{\alpha\beta} = -\frac{1}{2\pi} \int d^2\sigma \sqrt{-\det \tilde{h}_{\alpha\beta}} \equiv S_{NG}$$

✓

Under $h_{\alpha\beta} \to h_{\alpha\beta} + \delta h_{\alpha\beta}$, $S \to S + \delta S$ with

$\delta S \equiv \frac{1}{2} \int d^2\sigma \sqrt{\det h}\, T^{\alpha\beta} \delta h_{\alpha\beta}$ so $h_{\alpha\beta}$ EOM, $\frac{\delta S}{\delta h_{\alpha\beta}} = 0$,

$\Rightarrow T_{\alpha\beta} = 0$. $\leftarrow$ EOM is <u>on shell</u> cond. for physical states.

First impose much weaker cond under const. scaling $h_{\alpha\beta} \to \lambda h_{\alpha\beta}$ Scale transformation with coords. σ^α held fixed. Alternatively, scale σ^α & leave $h_{\alpha\beta}$ fixed. ~~[illegible]~~ $ds^2 = h_{\alpha\beta} d\sigma^\alpha d\sigma^\beta$ not invt., not just a reparametrization symm. Conformal symm.

$\Rightarrow$ correct vacuum of thy.

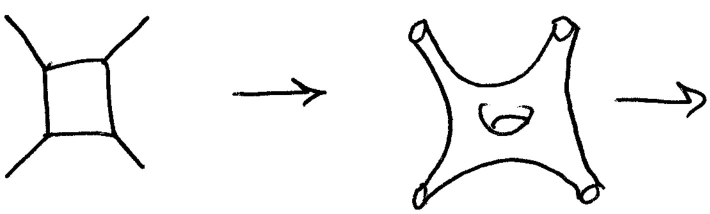
E.g. taking $S = -\frac{1}{2\pi\alpha'}\int d^2\sigma \; g_{\mu\nu}(X)\partial_\alpha X^\mu \partial^\alpha X^\nu$

($= S_{\text{polyakov}}$ with $h_{\alpha\beta} = \eta_{\alpha\beta}$)

Condition of scale inv at quantum level will $\Longrightarrow$ Einstein's eqns (with stringy corrections) for spacetime metric $g_{\mu\nu}$.

Get spacetime physics from 2d QFT on the worldsheet. Spacetime EOM related to conformal symm of world sheet thy.

Spacetime amplitudes via Feynman diagram.



$\rightarrow$ Correlation function of operators. in 2d thy.

Spacetime states $\rightarrow$ Operators in 2d worldsheet thy inserted at places $\rightarrow$  .

study Scale Invariance in QFT. Thy invt under scaling $g_{\mu\nu} \to \lambda g_{\mu\nu}$ with x^μ invt (or could scale x^μ with g invt.)

$\Rightarrow$ $\delta S = 0$ for $\delta g_{\mu\nu} = g_{\mu\nu} \delta\lambda$

$$0 = \frac{1}{2} \int d^D x \sqrt{-\det g}\, T^{\mu\nu} g_{\mu\nu} \delta\lambda$$

$\Rightarrow$ $T^{\mu\nu} g_{\mu\nu} = 0 \Rightarrow \boxed{T_\mu{}^\mu = 0}$ traceless stress tensor. Condition for scale inv. in any spacetime dimension D. $\Rightarrow$ Conserved dilitation current $D^\mu = T^{\mu\nu} x_\nu$.

~~[illegible]~~ In D spacetime dimensions, scale inv. $\Rightarrow$ Symmetry group with finite number of generators. However in 2dim. the symmetry group has infinite # of generators. Related to why in 2d can do Weyl scaling $h_{\alpha\beta} \to f(\sigma) h_{\alpha\beta}$ by arb. function (not just const.) & eliminate "Liouville" field $\phi(\sigma)$ in $h_{\alpha\beta} = e^\phi \eta_{\alpha\beta}$.

Useful to study general aspects of 2d scale inv. theories. So spend some time in 2d with occasional reminders about string thy motivations.

2d Conformal Field Theory:

Helps to use complex coordinates $z, \bar{z}$ for 2d

Eg $\quad z \equiv \sigma^1 - \sigma^0 \qquad \bar{z} \equiv \sigma^1 - \sigma^0 \qquad$ (Mink)

$\quad z \equiv \sigma^1 + i\sigma^2 \qquad \bar{z} \equiv \sigma^1 - i\sigma^2 \qquad$ (Euc.)

Metric $\eta_{\alpha\beta} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ Mink or $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ Euc

$\hookrightarrow \; ds^2 = dz\, d\bar{z} \equiv |dz|^2 \rightarrow$

$$\eta_{z\bar{z}} = \frac{1}{2} \qquad \eta_{zz} = \eta_{\bar{z}\bar{z}} = 0$$

$$\eta^{\bar{z}z} = 2 \qquad \eta^{zz} = \eta^{\bar{z}\bar{z}} = 0$$

Vectors have 2 compts with upper or lower indices : $V^z, V^{\bar z}$ or $V_z, V_{\bar z}$

E.g. $V^z = dz \,, \quad V^{\bar z} = d\bar z$

$$V_z = \frac{\partial}{\partial z} \equiv \bar\partial \,, \qquad V_{\bar z} = \frac{\partial}{\partial \bar z} \equiv \bar\partial.$$

General coord transformations: $z \to f(z, \bar z)$

$\bar z \to \bar f(z, \bar z)$

for general functions f & $\bar f$.

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar z} d\bar z$$

$$d\bar f = \frac{\partial \bar f}{\partial z} dz + \frac{\partial \bar f}{\partial \bar z} d\bar z.$$

More generally $V^f = \frac{\partial f}{\partial z} V^z + \frac{\partial f}{\partial \bar z} V^{\bar z}$

$$V^{\bar f} = \frac{\partial \bar f}{\partial z} V^z + \frac{\partial \bar f}{\partial \bar z} V^{\bar z}$$

How we write $V'^\alpha = \left(\frac{\partial \sigma'^\alpha}{\partial \sigma^\beta} \right) V^\beta$ in the $z, \bar z$ coordinates.

Likewise $$\frac{\partial}{\partial f} = \frac{\partial z}{\partial f}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial f}\frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial \bar{f}} = \frac{\partial z}{\partial \bar{f}}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial \bar{f}}\frac{\partial}{\partial \bar{z}}$$

& more generally $$V_f = \frac{\partial z}{\partial f} V_z + \frac{\partial \bar{z}}{\partial f} V_{\bar{z}}$$

$$V_{\bar{f}} = \frac{\partial z}{\partial \bar{f}} V_z + \frac{\partial \bar{z}}{\partial \bar{f}} V_{\bar{z}}.$$

General tensors transform similarly. E.g.

$$g_{ff} = \left(\frac{\partial z}{\partial f}\right)^2 g_{zz} + \left(\frac{\partial \bar{z}}{\partial f}\right)^2 g_{\bar{z}\bar{z}} + \left(\frac{\partial z}{\partial f}\right)\left(\frac{\partial \bar{z}}{\partial f}\right) g_{z\bar{z}}.$$

<u>Special case</u>: $z \to f(z)$ with $\frac{\partial f}{\partial \bar{z}} = 0$

f = analytic & $\bar{z} \to \bar{f}(\bar{z})$ with $\frac{\partial \bar{f}}{\partial z} = 0$

$\bar{f}$ = anti analytic. Conformal mapping, preserves angles. Maps $ds^2 = |dz|^2 \to \left|\frac{dz}{df}\right|^2 |df|^2$

i.e. $$\left.\begin{aligned} \eta_{ff} &= \eta_{\bar{f}\bar{f}} = 0 \\ \eta_{f\bar{f}} &= \left|\frac{dz}{df}\right|^2 \eta_{z\bar{z}} = \frac{1}{2}\left|\frac{dz}{df}\right|^2 \end{aligned}\right\}$$ preserves metric up to Weyl rescaling by $\left|\frac{dz}{df}\right|^2$ factor

General scale inv. cond: $T_\alpha{}^\alpha = 0$. In 2d

$$\Rightarrow T_z{}^z + T_{\bar z}{}^{\bar z} = 0 \quad \Rightarrow \quad \boxed{T_{z\bar z} = 0}$$

Energy & mom. conservation $\partial_\mu T^{\mu\nu} = 0$

$$\Rightarrow \partial_z T_{\bar z z} + \partial_{\bar z} T_{zz} = 0$$

$$\partial_z T_{\bar z \bar z} + \partial_{\bar z} T_{z\bar z} = 0$$

use $T_{z\bar z} = 0$

$$\Rightarrow \partial_{\bar z} T_{zz} = 0 \qquad \text{write } \bar\partial T = 0$$

$$\partial_z T_{\bar z\bar z} = 0 \qquad \text{write } \partial \bar T = 0$$

Note that $\partial_{\bar z}\left[f(z) T_{zz}\right] = 0$ for any analytic $f(z)$. Special prop. of 2d: scale inv $\Rightarrow$ infinite # of conserved charges!

Charges:

$$L_m = \oint \frac{dz}{2\pi i} z^{m+1} T_{zz}$$

$m = -\infty \cdots +\infty$

$$\bar L_m = \oint \frac{d\bar z}{2\pi i} \bar z^{m+1} T_{\bar z\bar z}$$

$$T_{zz} = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \qquad \bar T_{\bar z\bar z} = \sum_{m=-\infty}^{\infty} \frac{\bar L_m}{\bar z^{m+2}}$$

$L_n, \bar L_n$ generators of $z \to f(z)$ $\bar z \to \bar f(\bar z)$

$$ds^2 = dz\,d\bar z \longrightarrow df\,d\bar f = \frac{\partial f}{\partial z}\frac{\partial \bar f}{\partial \bar z}\, ds^2 \quad \text{(not invt)}$$

Under $z \to z + \varepsilon(z)$ $\quad \bar{z} \to \bar{z} + \bar{\varepsilon}(\bar{z})$ for inf $\varepsilon, \bar{\varepsilon}$

general op $\Phi \to \Phi + \delta_{\varepsilon\bar{\varepsilon}} \Phi$ with

$$\delta_{\varepsilon\bar{\varepsilon}} \Phi(w, \bar{w}) = \oint \frac{dz}{2\pi i} \varepsilon(z) \left[T(z), \Phi \right]$$

$$+ \oint \frac{d\bar{z}}{2\pi i} \bar{\varepsilon}(\bar{z}) \left[\bar{T}(\bar{z}), \Phi(w, \bar{w}) \right].$$

"Primary" operators: $\Phi(z, \bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f, \bar{f})$

under $z \to f(z)$ $\quad \bar{z} \to \bar{f}(\bar{z})$. $\quad \Phi (dz)^h (d\bar{z})^{\bar{h}}$ inv't.

Not all ops are primary.

* If Φ is primary, find how $\partial_z \Phi$ transfs.

For primary Φ with $f = z + \varepsilon$ $\quad \bar{f} = \bar{z} + \bar{\varepsilon}$

$$\delta_{\varepsilon\bar{\varepsilon}} \Phi = \left(h(\partial\varepsilon) + \varepsilon\partial + \bar{h}(\bar{\partial}\bar{\varepsilon}) + \bar{\varepsilon}\bar{\partial} \right) \Phi(z, \bar{z}).$$

The stress tensor itself is generally not quite primary:

$$T(z) \to \left(\frac{\partial f}{\partial z}\right)^2 T(f(z)) + \frac{c}{12} \{f, z\}$$

$\{f, z\}$: Schwarzian deriv. $= \dfrac{2\partial_z^3 f \,\partial_z f - 3\partial_z^2 f \,\partial_z^2 f}{2\partial_z f \,\partial_z f}$

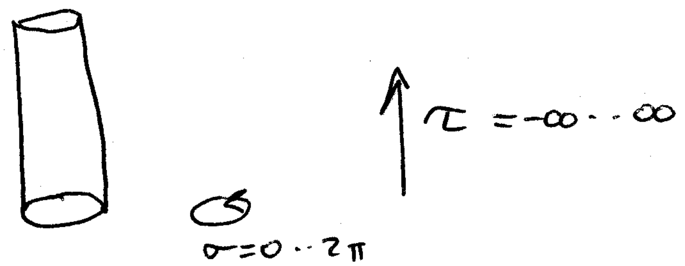
T is almost $h = 2$ $\bar{h} = 0$ primary.

For $f = z + \varepsilon(z)$ with ε inf. $T \to T + \delta_\varepsilon T$

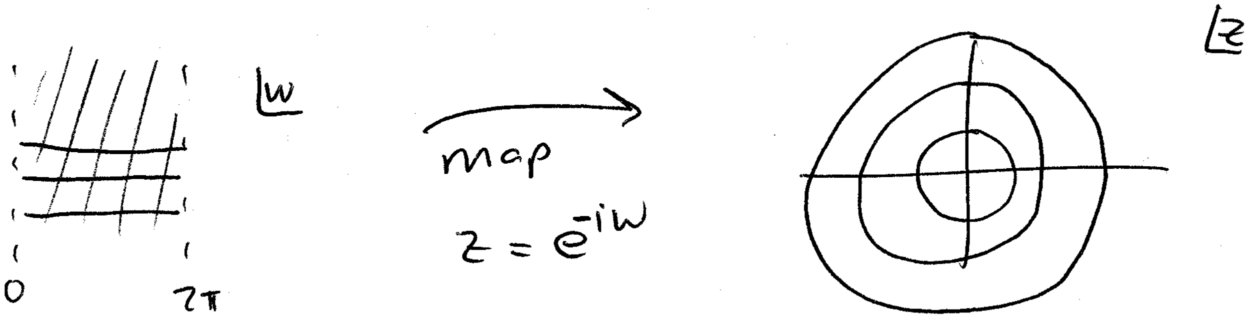
with $\delta_\varepsilon T(z) = \left(2(\partial_z \varepsilon) + \varepsilon \partial_z\right) T + \frac{c}{12} \partial^3 \varepsilon(z)$

c is the "central charge" or "conformal anomaly" of theory = real number specific to theory.

String moving in time $\to$ cylinder



let $w = \sigma + i\tau$
$\sim w + 2\pi$



$z = e^{-iw}$

equal worldsheet τ time $\longrightarrow$ equal radius in z plane

radial quantization.

$$T_{ww} = \left(\frac{\partial z}{\partial w}\right)^2 T_{zz} + \frac{c}{12}\{z, w\}$$

$$= -z^2 T_{zz} + \left(\frac{c}{12}\right)\left(+\frac{1}{2}\right)$$

$$T_{ww} = -\sum_{m=-\infty}^{\infty} T_m \left(e^{-iw}\right)^m \qquad T_{zz} = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}$$

Fourier modes

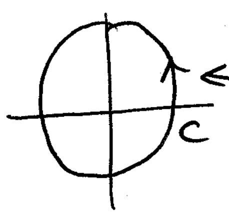
$$T_m = L_m - \delta_{m,0}\frac{c}{24}, \qquad \bar{T}_m = \bar{L}_m - \delta_{m,0}\frac{\bar{c}}{24}$$

Hamiltonian: $H = \int_0^{2\pi} \frac{d\sigma}{2\pi} T_{\tau\tau} = T_0 + \bar{T}_0$

$$\boxed{H = L_0 + \bar{L}_0 - \frac{(c+\bar{c})}{24}.}$$

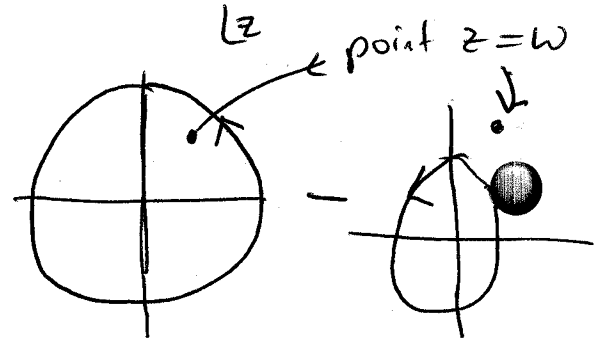
Casimir Energy
$E = -\frac{(c+\bar{c})}{24} \Rightarrow -\frac{\pi(c+\bar{c})}{12\ell}$ (ℓ = length)

Conserved charges $\qquad Q = \oint_C \frac{dz}{2\pi i} j, \quad \bar{Q} = \text{c.c.}$



contour ~ "equal time slice"

Commutator $\quad [Q, \mathcal{O}(w)] =$



radial ordering $\quad Q\,\mathcal{O}(w) \Rightarrow$ Q contour is outside $\mathcal{O}$ insertion point w, ie $|z| > |w|$

like time ordered prod.

$\mathcal{O}(w)\,Q \Rightarrow |w| > |z|.$

Commutator is difference =