215c, 1/30/21 Lecture outline. © Kenneth Intriligator 2021.

⋆ **Week 1 recommended reading: Schwartz sections 25.1, 28.2.2. 28.2.3. You can find more about group theory (and some of my notation) also here https://keni.ucsd.edu/s10/**

• At the end of 215b, I discussed spontaneous breaking of global symmetries. Some students might not have taken 215b last quarter, so I will briefly review some of that material. Also, this is an opportunity to introduce or review some group theory.

• Physics students first learn about continuous, non-Abelian Lie groups in the context of the 3d rotation group. Let $\vec{v}$ and $\vec{w}$ be N-component vectors (we'll set $N = 3$ soon). We can rotate the vectors, or leave the vectors alone and rotate our coordinate system backwards – these are active vs passive equivalent perspectives – and scalar quantities like $\vec{v} \cdot \vec{w}$ are invariant. This is because the inner product is via δ_{ij} and the rotation is a similarity transformation that preserves this. Any rotation matrix R is an orthogonal $N \times N$ matrix, $R^T = R^{-1}$, and the space of all such matrices is the $O(N)$ group manifold. Note that $\det R = \pm 1$ has two components, and we can restrict to the component with $\det R = 1$, which is connected to the identity and called $SO(N)$. The case $SO(2)$, rotations in a plane, is parameterized by an angle $\theta \cong \theta + 2\pi$, so the group manifold is a circle. If we use $z = x + iy$, we see that $SO(2) \cong U(1)$, the group of unitary 1×1 matrices $e^{i\theta}$. The case $SO(3)$ is parameterized by 3 angles – the Euler angles, e.g. we can rotate $\hat{z}$ to some $\hat{n}$ specified by the usual θ and ϕ of polar coordinates, and then rotate by a 3rd angle ξ around $\hat{n}$. In QM we learn that we generally need a double cover of the rotation group to allow for spinors where a 2π rotation is -1 rather than $+1$. This version of the rotation group is $SU(2)$, the group of unitary 2×2 matrices $U^\dagger = U^{-1}$ with $\det U = 1$. If we're not making the distinction about global issues we can roughly say that $SO(3) \cong SU(2)$; for higher N the groups $SO(N)$ and $SU(N)$ are distinct also locally. Note that $SO(N)$ rotations can be written as $R = e^{i\phi^a T^a}$ where T^a is an antisymmetric $N \times N$ matrices, so there are $\frac{1}{2}N(N-1)$ of them, i.e. the index $a = 1 \ldots \frac{1}{2}N(N-1)$. The group $SU(N)$ has $U = e^{i\phi^a T^a}$ with T^a Hermitian and traceless, so $a = 1 \ldots N^2 - 1$. The T^a are called the group generators. For $SU(2)$, we are very familiar with this from QM, where $T^a = J^a/\hbar$, is the angular momentum. Groups have a multiplication rule, corresponding to multiplication of the matrices R or U, and a group is non-Abelian if some $gh \neq hg$. Writing $g = e^{i\phi^a_g T^a}$, then $[T^a, T^b] = i f_{abc} T^c$, and the group is non-Abelian if the Lie algebra structure constants $f_{abc} \neq 0$. The fact that group multiplication must give another group element requires the

Jacobi identity for the commutators, imposing conditions on the structure constants f_{abc}; $SU(2)$ is the simplest non-Abelian Lie group, with $f_{abc} = \epsilon_{abc}$.

The Lorentz group is $SO(1,3)$, which is an example of a non-compact group since the Lorentz boost parameter (e.g. the rapidity) lives on the non-compact real line rather than a circle. If we continue to Euclidean space, then we instead get $SO(4) \cong SU(2)_L \times SU(2)_R$.

• Let's now recall the notation of the *representation* of the group. In the case of $SU(2)$, recall that we write $|j,m\rangle$ with $m = -j, \dots j$ and j integer or half-integer. Here j labels the representation, which is $(2j+1)$-dimensional, and m runs over the dimension of the representation. If we write $\langle jm'|J^a/\hbar|jm\rangle = (T^a_j)_{m',m}$, then that is the generator's matrix element in that representation; it satisfies the Lie algebra commutation relations and we can exponentiate it to get the group elements in that representation. The case $j = 0$ is the trivial representation, where we replace $T^a \to 0$ and $g \to 1$. This is a one-dimensional representation. The case $j = \frac{1}{2}$ is a two-dimensional representation, and it is called the fundamental representation; in the fundamental representation, the group elements are taken to be literally the $SU(2)$ matrices themselves. It is called fundamental because taking tensor products with the fundamental representation leads to all representations; recall addition of angular momentum. The case $j = 1$ is called the adjoint representation; the special thing about the adjoint representation is that its dimension is the same as the number of generators (three for $SU(2)$), and indeed for any Lie algebra the adjoint representation has $(T^a)^b{}_c = f^{ab}_c$. The Jacobi identify ensures that this satisfies the Lie algebra's commutation relations. The other j representations do not have special names.

• Now consider $SU(3)$. If we write $U = e^{i\phi^a T^a}$, then $a = 1 \dots 8$, and the 8 generators have $[T^a, T^b] = if^{ab}_c T^c$ for some structure constants f^{ab}_c. For the case of $SU(2)$, the generators in the fundamental representation are $T^a = \frac{1}{2}\sigma^a$, where we call σ^a the Pauli matrices. Likewise for $SU(3)$, we can write $T^a = \frac{1}{2}\lambda^a$, where we call the λ^a the Gell-Mann matrices. Particle physicists first met $SU(3)$ in the context of what I'll call $SU(3)_F$, an approximate global symmetry (with F for flavor) that rotates the (u,d,s) quark flavors into each other. Later, it was learned that the strong force is given by a conceptually completely different instance of $SU(3)$: the $SU(3)_C$ gauge symmetry that acts on (r,g,b) colors of quarks. I will first say some general things about any instance of $SU(3)$, and then illustrate it in the context of $SU(3)_F$. Then $SU(3)_C$ will be discussed a bit later.

The λ^a are 3×3 matrices, and $T^a = \frac{1}{2}\lambda^a$ is the fundamental representation of $SU(3)$. For general $SU(N)$, every representation has an anti-version, where we replace $U \to U^*$, i.e. $T^a \to -(T^a)^*$. For the special case of $SU(2)$, this is not interesting because they differ by a similarity transformation, and $SU(2)$ is thus called pseduoreal. For $SU(N>2)$ the rep and its anti-rep can be inequivalent; in particular there is both a fundamental and an anti-fundamental representation, and they differ from each other. They are often called the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations. The adjoint representation, where $T^a \to f_c^{ab}$ is called the $\mathbf{8}$ and it is said to be real because $\mathbf{8} \cong \bar{\mathbf{8}}$. Examples of $SU(3)$ tensor products are $\mathbf{3}\times\mathbf{3} = \bar{\mathbf{3}}_A + \mathbf{6}_S$ and $\mathbf{3}\times\bar{\mathbf{3}} = \mathbf{1}+\mathbf{8}$. Note that the last one shows that $\mathbf{1}$ and $\mathbf{8}$ are real and it is similar to what in $SU(2)$ notation where reps are labeled by (j) we'd write as $(\frac{1}{2})\times(\frac{1}{2}) = (0)_A + (1)_S$, which if we label instead by the dimension of the rep we'd write as $\mathbf{2}\times\mathbf{2} = \mathbf{1}+\mathbf{3}$. The subscripts are useful if the two reps on the LHS are the same, and refer to antisymmetric vs symmetric under exchange of the two.

Note that $\mathbf{3}\times\mathbf{3}\times\mathbf{3} = \mathbf{1}_A + \ldots$, where the $\mathbf{1}_A$ is completely antisymmetric in all three objects. Let u^i, v^i, and w^i denote the three fundamental objects on the LHS. They transform as $u^i \to U^i{}_j u^j$ under a $SU(3)$ transformation. The statement is that $u^i u^j u^k \epsilon_{ijk}$ is $SU(3)$ invariant. This is because ϵ_{ijk} transforms to itself with a factor of $\det U$, and the S in $SU(3)$ is because we impose $\det U = 1$. Likewise, the $\mathbf{1}$ in $3\times\bar{3}$ can be understood as the statement that $\bar{u}_i v^i \to \bar{u}U^\dagger U v = \bar{u}v$ because $U^\dagger U = 1$, which can be understood in terms of δ_i^j being an $SU(3)$ invariant tensor.

• We know from QM that it's useful to go to a basis where we diagonalize all of the commuting observables. Similarly, for group theory we can choose to diagonalize as many of the T^a as possible. For $SU(2)$, we can diagonalize one of the three generators, and we choose to call that one J_3. For $SU(3)$, we can diagonalize two of the eight generators and we choose to call those T^3 and T^8, with $\lambda_3 = diag(1,-1,0)$ and $\lambda_8 = \frac{1}{\sqrt{3}} diag(1,1,-2)$. The fact that λ_8 and $-\lambda_8$ have different eigenvalues illustrates that the $\mathbf{3}$ and $\bar{\mathbf{3}}$ reps are inequivalent. For $SU(2)$, the values m run over what is called the weights of the representation, for the fundamental representation the weights are $\pm\frac{1}{2}$ and for the adjoint they are $-1,0,1$. For $SU(3)$, the weights are a 2d vector, corresponding to the T^3 and T^8 eigenvalues. For the fundamental representation, the weights are $(\frac{1}{2},\frac{1}{2\sqrt{3}})$, $(-\frac{1}{2},\frac{1}{2\sqrt{3}})$, and $(0,-\frac{1}{\sqrt{3}})$, forming a little equilateral triangle. All $SU(3)$ weights can be obtained by composing this triangle to form other triangles and shapes. The weights of the anti-fundamental representation are minus these, corresponding to a reflected version of the basic triangle. We can write raising and lowering operators $E_{\pm 1,0} = \frac{1}{\sqrt{2}}(T_1 \pm iT_2)$ in analogy with the $J_\pm$ of $SU(2)$: these raise or lower the first weight by 1 unit, and do not affect the second weight. $SU(3)$ symmetry relates them to $E_{\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{2}}(T_4 \pm iT_5)$ and

$E_{\mp\frac{1}{2},\pm\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{2}}(T_6 \pm iT_7)$. Each of these must either map a weight to another weight, or they act to annihilate that weight. The subscripts of the E's give the six non-zero weights of the adjoint representation, forming a hexagon, with T^3 and T^8 corresponding to two weights of $(0,0)$ since they commute with each other.

• Let's illustrate this with some physics. Recall that the Dirac Lagrangian naturally admits unitary symmetries that act on the Fermion ψ. Consider $\mathcal{L} = \sum_{j=1}^{N} \bar{\psi}_j(i\not{D} - m_j)\psi_j$. If the m_j are all equal, there is a $U(N)$ symmetry. If the m_j are all zero, it enhances to an $U(N)_L \times U(N)_R$ symmetry, acting on $\psi_{L,R} = \frac{1}{2}(1 \mp \gamma_5)\psi$. Recall that in the Dirac basis $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so $\psi_{L,R}$ can be written as 2-component Fermions. The mass terms transform as a bi-fundamental under $U(N)_L \times U(N)_R$, and if it is proportional to the identity matrix then it explicitly breaks $U(N)_L \times U(N)_R \to U(N)_{D=L+R}$. Write $U(N)_L \times U(N)_R \cong SU(N)_L \times SU(N)_R \times U(1)_V \times U(1)_A$. The currents are $j_V^\mu = \sum_j \bar{\psi}_j \gamma^\mu \psi_j$, while $j_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi_j$, and the $SU(N_f)_{L,R}$ currents are $j_{L,R}^a = \bar{\psi}T^a P_{L,R}\psi$. We will later discuss anomalies and see that $U(1)_A$ can be violated by quantum effects.

I now have a decision to make: which symmetry should I illustrate first. There are several options. In the Standard Model (advertisement for this week's colloquium by Steven Weinberg!), there are quarks in three families $\begin{pmatrix} u \\ d \end{pmatrix}$, $\begin{pmatrix} c \\ s \end{pmatrix}$, $\begin{pmatrix} t \\ b \end{pmatrix}$. Each of these doublet matrices denotes a $\mathbf{2}$ fundamental representation of $su(2)_W$ gauge symmetry (here W stands either for Weak or Weinberg). All 6 quark "flavors" are in the $\mathbf{3}$ of the $su(3)_C$ gauge symmetry of the strong force, which rotates a "color" gauge index e.g. $(red, green, blue)$. Baryons (e.g. the proton, neutron, and many exotics) are made up from 3 quarks, which are in the color singlet component of $\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{1}_A + \ldots$. Mesons (e.g. the pions) are made up from a quark and an anti-quark, which are in the color singlet component of $\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + \ldots$. These illustrate the fact that $su(3)_C$ is in a confining phase.

I will discuss non-Abelian gauge theories, such as $su(2)_W$ and $su(3)_C$ soon. But first I will follow the historical route and illustrate $SU(3)$ group theory via an approximate $SU(3)_F$ global symmetry that rotates the (u,d,s) quark flavors. This is an enhancement of the $SU(2)_{isospin}$ that was found to be an approximate symmetry of nuclear physics, with e.g. $\begin{pmatrix} p \\ n \end{pmatrix}$ a doublet and (π^+, π^0, π^-) a triplet. In terms of the underlying quarks, isospin is the $SU(2)_F \subset SU(3)_F$ that rotates the (u,d) quarks. The up and the down and strange quarks are pretty light, so we can approximate their mass as zero to get a pretty good approximate. They have different electric and $SU(2)_W$ charges, but those are small effects compared with the strong force – as far as the strong force is concerned, they are all the same. So as far as the strong force is concerned there is an approximate $SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$ classical global symmetry. The qualifier classical was included as a legal disclaimer, because $U(1)_A$ is actually violated by the strong force, by the Adler-Bell-Jackiw anomaly, to be discussed a bit later. So the actual (approximate) global symmetry is $G = SU(3)_L \times SU(3)_R \times U(1)_V$.

The above (approximate) global symmetry turns out to be spontaneously broken by the vacuum, $\langle\bar{\psi}\psi\rangle \sim \Lambda^3_{QCD}$ (with $F_\pi \approx 190MeV$ and $\Lambda_{QCD} \approx 300MeV$) to the subgroup $H = SU(3)_{D=L+R} \times U(1)_V$. Recall from last quarter Goldstone's theorem: whenever a continuous global symmetry G is broken by the vacuum to a subgroup H, then the broken currents do not annihilate the vacuum but instead create a massless scalar field particle: $\langle\pi^a(p)|j^{\mu,a}_{G/H}(x)|\Omega\rangle = iF_\pi p^\mu e^{-ip\cdot x}$, where the π_a take values in the compact coset space G/H. It is usually written as $U(x) \in SU(N_f) = \exp(\frac{2i}{f_\pi}\pi(x))$ with $\pi(x) \equiv \pi^a(x)T^a$. Although the $SU(3)$ is a pretty good approximation, an $SU(2)$ subgroup is even better, because the strange quark isn't as light as the up and down quarks. If we focus on the subgroup $SU(2)_L \times SU(2)_R \to SU(2)_{D=L+R}$, the $G/H \cong SU(2) \cong S^3$, our Nambu Goldstone bosons take values in a 3-sphere. The remaining $SU(2)_D$ is called isospin, and all particles can be organized into isospin representations. The NGBs transform as $\pi \to g_L\pi g_R^\dagger$, where $g_{L,R} \in SU(2)_{L,R}$, and this corresponds to the adjoint representation under $SU(2)_D$. Ignoring the quark masses and electromagnetism and the weak force, there would be three massless NGB scalar particles, and these are identified with the pions $\pi^\pm$ and π^0. They are indeed quite light, and their masses can be viewed as coming from the underlying quark masses (more precisely, from the Yukawas coupling them to the Higgs field) along with effects from the electroweak force: find $m_u \approx 2MeV$, $m_d \approx 4.8MeV$. The masses of the three lightest pions are $m_{\pi^0} = 135MeV$ and $m_{\pi^\pm} = 140MeV$. The π^+ contains $u\bar{d}$ quarks and has a lifetime of $\sim 10^{-8}s$ ($\pi^+ \to \bar{\mu} + \nu_\mu$), and π^0 contains $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$ quarks with a lifetime of $10^{-16}s$ ($\pi^0 \to 2\gamma$).

If we include the strange quark then there are 8 light pions, corresponding to the adjoint representation of $SU(3)_{D=L+R}$:

$$\begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{-\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -2\frac{\eta^0}{\sqrt{6}} \end{pmatrix} \in \mathbf{8}.$$

We can plot the quarks and these mesons in the (T_3, T_8) plane, with the quarks in a triangle and the above mesons in a hexagon. It is conventional to plot it in terms of Q_{elec} and S, and also hypercharge $Y = T_8(2/\sqrt{3}) \equiv B + S$ and $Q_{elec} = T_3 + Y/2$. The top row of the hexagon has the $K^0 and K^+$, with $s = 1$. The next row has the π^- and π^+ at the ends, and π^0 and η in the middle, with $s = 0$. The next row has the K^0 and $\bar{K}^0$ on the bottom, with $s = -1$. Note that electric charge is constant along the diagonals, with $q = -1$, $q = 0$, and $q = 1$. This is the original "Eightfold way" of Gell-Mann.

The baryons, including the proton and the neutron and others, also form $SU(3)_F$ representations and Gell-Mann used e.g. $SU(3)_F$ to predict the existence, and the mass, of a then unseen but later experimentally confirmed baryon that is now understood to be made up from three strange quarks in the **10** of $SU(3)_F$ and with spin $j = 3/2$. Note that this is completely symmetric in the $SU(3)_F$ labels and in the spin, and this fits with Fermi statistics because it is completely antisymmetric in $su(3)_c$ to get something color neutral.

• Write $U(x) \in SU(N_f) = \exp(\frac{2i}{f_\pi}\pi(x))$ with $\pi(x) \equiv \pi^a(x)T^a$. E.g. for $SU(2)$ the π^a are the Euler angles, which live in a compact space and parameterize a $S^3 \cong SU(2)$. The broken $SU(N_f)_L \times SU(N_f)_R$ act on U as $U \to L^\dagger U R$ and using the unbroken $SU(N_f)_D$ we can locally rotate $U \to 1$ and see that is preserved if $L = R$. Note that $\mathrm{Tr}(U^\dagger \partial_\mu U) = 0$ because the generators are traceless, and $U^\dagger \partial_\mu U = -\partial U^\dagger U$, so there is a unique kinetic term that respects the symmetry (called the chiral Lagrangian):

$$\mathcal{L}_2 = \frac{f_\pi^2}{4}\mathrm{Tr}(\partial^\mu U^\dagger \partial_\mu U) = \tfrac{1}{2}\partial^\mu \pi^a \partial_\mu \pi^a - \frac{2}{3f_\pi^2}\mathrm{Tr}(\pi^2(\partial\pi)^2 - (\pi\partial\pi)^2) + \ldots.$$

This describes the massless pions, and their derivative interactions. This is the low-energy effective field theory for the spontaneously broken phase, and the theory can be treated as a LEEFT with a cutoff at $\sim f_\pi \sim v \sim \Lambda$. We can see how aspects of the original, microscopic theory show up in the LEEFT dual.

The $SU(N_f)_D$ global symmetry is manifest and unbroken in the LEEFT. The separate $SU(N_f)_L$ and $SU(N_f)_R$ secret symmetries act as $U \to L^\dagger U R$ and are thus realized as shifts of the NGBs. The $SU(N_f)_L$ current acts as $\delta_L U \approx -i\alpha^a T^a U \approx -\frac{1}{2} f_\pi \partial_\mu \pi^a$, which fits with the fact that these currents do not annihilate the vacuum but instead act on it to create the NBGs, $\langle \pi^b(p)|J^a_{L,\mu}(x)|0\rangle = i\frac{f_\pi}{2}\delta^{ab}p_\mu e^{ipx}$. Likewise $J^a_{R,\mu} \approx +\frac{1}{2} f_\pi \partial_\mu \pi^a$ so the diagonal sum is unbroken.

Parity P takes $\vec{x} \to -\vec{x}$ and thus $\gamma_5 \to -\gamma_5$ and thus exchanges $L \leftrightarrow R$. It thus takes $U \to U^*$ so $\pi^a \to -\pi^a$. This fits with $\partial_\mu \pi^a \sim J^a_{L-R,\mu}$. This shows that pions transform as parity odd pseudoscalars, which fits with observation (based on which decays are allowed vs suppressed).

If we add mass terms $\delta_{\mathcal{L}} = m_{i\tilde{j}}\psi^i\tilde{\psi}^{\tilde{j}} + h.c.$, we explicitly break $SU(N_f)_L$ and $SU(N_f)_R$ because the masses are in the $(\bar{N}_f, N_f)$. This shows up in the LEEFT as mass terms for the NGBs: $\delta_{\mathcal{L}} = \frac{1}{2}\sigma \mathrm{Tr} mU + h.c. \to -\frac{\sigma}{f_\pi^2}\mathrm{Tr}(m + m^\dagger)\pi^2 + \ldots$. Note that the mass-squared of the NGBs is proportional to the Fermion mass times the SSB vev.

• If $U(1)_A$ were a symmetry, there would have to be a 9th pseudoscalar (since it is P odd) meson; the candidate observed particle is called the η', but it is too massive to be considered an approximate NGB. The resolution is that $U(1)_A$ is not a symmetry, as already mentioned, and this gives the η' a large mass compared to the light pions. The pions are not massless because the global symmetries are explicitly broken by the non-zero quark mass terms; approximate values are $m_u \approx m_d \approx 0.307 GeV$, $m_s \approx 0.490 GeV$, and approximate formulae for the meson masses from this explicit breaking would give $m_{\eta', wrong} \approx 355 MeV$ whereas $m_{\eta', actual} \approx 958 MeV$.

• The G/H space $SU(N_f)_D$ has non-trivial topology: it contains a S^3 so $\pi_3(G/H) = \mathbf{Z}$ for $N_f \geq 2$. For $N_f \geq 3$ it also contains a S^5, so $\pi_5(G/H) = \mathbf{Z}$ for $N_f \geq 3$. The S^3 means that there can be solitonic particle configurations, where space and the point at infinity, are wound around the S^3 target; these are called Skyrmions, and it turns out that they have the right quantum numbers to give the baryons of the original UV theory of quarks and gluons, now realized as solitons on the space of pions. The S^5 plays a role in giving what is known as the Wess-Zumino-Witten interaction of the low-energy theory. If there is time, this will be discussed in the context of 't Hooft anomaly matching.