215c, 4/27/20 Lecture outline.

⋆ **Week 5 reading: Tong chapter 2.**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• Briefly mention pair creation in an external electric field. E.g. Schwinger calculated in 1951 that the rate per volume for pair creating particles of mass m and charge q in an external electric field E is $\Gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(qE)^2}{(2\pi)^3 n^2} e^{-\pi m^2 n/|qE|}$. Need a big eE – not yet observed. Schwinger's calculation was to consider one-loop of virtual e^+e^- connecting with gauge field propagators to arbitrary $2n$-point functions of the external gauge field. Affleck, Alvarez, and Manton (1982) found a nice rederivation of this result using instantons and the worldline path integral for the charged particle in an external field. In Euclidean space we can take $A_\mu^{ext} = \frac{1}{2} F_{\mu\nu}^{ext} x^\nu$ with $F_{34}^{ext} = iE$. The electric field in Euclidean space resembles a magnetic field and the classical motion in a constant electric field $\vec{E} = E\hat{z}$ is circular, similar to a cyclotron orbit in the (x_3, x_4) Euclidean plane, with radius $m/|qE|$. The classical Euclidean action of the n-instanton configuration, where it rotates around n times, is then computed to be $S = n\pi m^2/|qE|$. Then find that Γ (including of course the prefactor) can be evaluated by summing over such instanton configurations. The solution represents tunneling between the vacuum with zero particles and the vacuum with a particle-antiparticle pair. This could also be a nice topic for a final presentation.

• Next topic: renormalization group running of g_{YM} and the Case of the Negative Beta Function. Recall the notions of the RG, where we integrate out UV modes above some scale μ and consider running as the scale is changed. Operators have scaling dimension Δ, which can include quantum corrections γ. In d spacetime dimensions, $\Delta(\mathcal{L}) = d$ and an operator $\mathcal{O}$ is relevant if $\Delta < d$, irrelevant if $\Delta > d$, and marginal if $\Delta = d$. If an irrelevant operator is added to $\mathcal{L}$, its effect becomes small in the IR, and large in the UV, and visa-versa for relevant operators. In $d = 4$ for example, mass terms like $m^2\phi^2$ and $m\bar{\psi}\psi$ are relevant. The couplings in $\lambda\phi^4$, and $h\phi\bar{\psi}\psi$, and the fine structure constant of QED $\alpha = e^2/4\pi\hbar c$ are all classically dimensionless, corresponding to the fact that they are coefficients of operators that are all classically marginal. In the quantum theory, these operators get anomalous dimension and correspondingly the operators get non-zero beta functions. This can be regarded as an anomaly: the theories with only these couplings (and no mass terms) have a classical $x^\mu \to \lambda x^\mu$ scale transformation, with Noether current the dilatation current $j_D^\mu = T^{\mu\nu} x_\nu$, which is classically conserved because classically $T_\mu^\mu = 0$. Quantum effects break this classical symmetry: $\partial_\mu j_D^\mu = T_\mu^\mu \sim \beta \mathcal{O}_{int}$, e.g. for gauge theory $T^\mu_\mu \propto \beta(g^{-2})F_{\mu\nu}F^{\mu\nu}$. "Dimensional transmutation ": the classically dimensionless couplings are replaced with a dynamically generated mass scale Λ. Recall that $\beta(g) \equiv \frac{dg}{d\ln\mu}$ and it turns out that all of the above-mentioned examples have $\beta > 0$, corresponding to the fact that the operators are all marginally irrelevant – they flow to zero in the IR. In fact, there is a proof (Coleman and Gross '73) that, without non-Abelian gauge theories, there cannot be marginally relevant couplings. E.g. $\lambda\phi^4$ has $\beta(\lambda) = 3\lambda^2/16\pi^2 + O(\lambda^3) > 0$. Integrating the one-loop term gives $\exp(-8\pi^2/\lambda(\mu)) \approx (\mu/\Lambda)^{3/2}$, where we need $\mu \ll \Lambda$ for perturbation theory to hold; Λ is a UV cutoff, and in the IR, as $\mu \to 0$, we see that $\lambda \to 0$, so the theory is IR free. QED is qualitatively similar – this is the Landau pole – as is Yukawa theory. For QED, the one-loop beta function is $\beta(\alpha) = \mathrm{Tr}(Q^2)\frac{2\alpha}{3\pi} + O(\alpha^2) > 0$, where $\mathrm{Tr}Q^2 \equiv \sum_{f=1}^{N_f} q_f^2$ (this is with charged Fermions - one could similarly compute for the case of charged scalars; we do not consider the case of charged spin 1 because we're interested here in massless fields and there is a problem, proved by Weinberg and Witten, with massless charged fields of spin $j > \frac{1}{2}$ coupling to conserved currents. A non-Abelian gauge field can be thought of as a massless spin 1 contribution, and the loophole is that it couples to a covariantly conserved current; this loophole is what leads to the negative beta function for YM) and integrating the one-loop beta function gives $\exp(-8\pi^2/e^2(\mu)) \approx (\frac{\mu}{\Lambda})^{4\mathrm{Tr}Q^2/3}$, where again Λ is a UV cutoff and the theory is IR free for $\mu \to 0$. Likewise the Yukawa coupling has $\beta(h) \geq 0$, where $\beta = 0$ for the free theory with zero coupling.In fact, for Yukawa theory in the context of nuclear physics is because this is a low energy approximation to QCD, so the UV cutoff is related to where the description breaks down.

• Draw pictures of RG flows and mention the history from the 1970s, where theorists were trying to kill off QFT as a sensible theory by showing Landau pole behavior in the UV. Mention asymptotic safety question. There were more and more general proofs that general unitary interacting theories will lead to $\beta > 0$. At that time, Yang-Mills theory was not considered to be important, so computing its beta function seems to have been largely a curiosity and grad student project to verify that it too leads to $\beta > 0$ – but there was a surprise! In hindsight, it has a loophole in the proofs that $\beta > 0$ because the gauge fields A^a_μ are in the adjoint representation rather than gauge invariant.

• Intuitive vacuum screening picture in QED: the positive beta function is from polarizing the vacuum, which screens electric charge. Like $\vec{D} = \epsilon\vec{E}$ with $\epsilon = \epsilon_0(1+\chi_e)$ with electric susceptibility $\chi_e > 0$, so polarization reduces the electric field. For magnetic field $\vec{B} = \mu\vec{H}$ with $\mu = \mu_0(1+\chi_m)$, with either sign χ_m (diamagnets if $-1 \leq \chi_m < 0$ and paramagnets if $\chi_m > 0$). The QFT vacuum can be polarized, but need $\epsilon\mu = 1$ for the gauge fields to move at $c = 1$. So diamagnetic vacuum has electric screening ($\beta > 0$) and paramagnetic vacuum has electric anti-screening ($\beta < 0$).

• We will sketch how to compute β_{1-loop} for Yang-Mills. It is negative, was computed by Gross and Wilczek, and independently by Politzer, and this was recognized by a Nobel prize in 2004. Picture of the RG flows, with asymptotic freedom in UV and strong coupling in the IR. Matches beautifully to real-world QCD, especially for $E \gg \Lambda_{QCD}$ where the theory is weakly coupled and precision calculations and comparison to experiment is possible. ($\alpha_s(M_Z) \approx 0.1$).

• To build up to that, let's write down the Feynman rules for the case of pure Yang-Mills. We can add charged matter later (the matter contributes positively to the beta function, so $\beta < 0$ only if there are not too many matter fields).

Consider first the gauge field propagator. As in QED, we can see it from the path integral perspective by considering the $A^a_\mu A^b_\nu$ terms in $\mathcal{L}$ and the Gaussian integral leads to the inverse of the differential operator that acts on them. As in QED, there is a subtlety from gauge invariance: the differential operator has zero modes from pure gauge configurations, and the procedure is to fix a gauge and then one can check that the results in the end are gauge invariant. If, as in QED, we add a $-\frac{1}{2\xi}(\partial_\mu A^{a,\mu})^2$ gauge fixing term, we get a gauge field propagator $i\delta^{ab}(-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2})/(p^2 + i\epsilon)$ and the results are independent of ξ. But things are a bit more complicated than in QED, related to the nonlinearities of YM, because the gauge fields are in the adjoint representation and the currents are covariantly conserved. The upshot is that we need to consider ghosts. Feynman and Bryce DeWitt (independently) first noticed this in the context of trying to quantize gravity in the 1960s. They realized that the non-linearity led to problems, and studied Yang-Mills theory, which at the time was just a mathematical possibility, as a toy model for gravity.