215c, 4/15/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 3 reading: Tong chapter 2.**

**http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html**

• Last time: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A^\mu - i[A_\mu, A_\nu]$ is in the adjoint representation of the gauge group, and we consider $\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{matter} + \mathcal{L}_\theta$ where $\mathcal{L}_{YM} = -\frac{1}{2g^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$, and $\mathcal{L}_{matter}$ depends on the theory's dynamical matter, e.g. it could be $\mathcal{L}_{Dirac} = \bar{\psi}(i\not{D} - m)\psi$.

• Recall an issue with quantizing gauge fields: $\Pi_0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^0)} = 0$, which is a constraint rather than something to quantize, and hence the $A_0$ EOM (Gauss' law) becomes a constraint on physical states. We can choose $A^0 = 0$ by a gauge choice. It helps to gauge fix, and then can verify at the end that the results are independent of the gauge choice. We will return to gauge fixing choices and issues more later.

• The theta term for a non-Abelian gauge theory is

$$S_\theta = \frac{\theta}{16\pi^2}\int d^4x \mathrm{Tr}\, {}^*F^{\mu\nu}F_{\mu\nu} = \frac{\theta}{8\pi^2}\int \mathrm{Tr} F \wedge F = \theta \int c_2(F),$$

${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ and $c(F) = \sum_n c_n(F) = e^{F/2\pi}$. In the $u(1)$ case it involved $c_1(F) \wedge c_1(F)$ but here it's $c_2(F)$ and $c_1(F) = 0$ since $\mathrm{Tr}T^a = 0$. Many aspects are similar to the $u(1)$ case. As in the $u(1)$ case, $S_\theta$ is topological in its dependence in spacetime: the Lorentz indices are contracted with an epsilon tensor, rather than the spacetime metric, and it can only depend on the spacetime via the topology. As in the $u(1)$ case, it is also topological in its dependence on the gauge field – this is because the $\theta$ term is a (sort of) total derivative:

$$S_\theta = \theta \int d^4x \partial_\mu K^\mu \qquad K^\mu = \frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(A_\nu \partial_\rho A_\sigma - \frac{2}{3} i A_\nu A_\rho A_\sigma),$$

In terms of differential forms

$$c_2(F) = \frac{1}{8\pi^2}\mathrm{Tr} F \wedge F = dCS(A), \qquad CS(A) = \frac{1}{8\pi^2}\mathrm{Tr}(AdA - \frac{2}{3} i A \wedge A \wedge A),$$

In the notation with anti-Hermitan generators, $F_{here} \to iF_{new}$ etc, the new quantities satisfy $c_2(F) = dCS(A)$ with $CS(A) = \frac{1}{8\pi^2}Tr(AdA + \frac{2}{3}A^3)$. As in the $u(1)$ case, the classical EOM are unaffected if $\theta$ is constant, and there would be corrections to the $A_\mu$ EOM involving $\partial_\mu\theta$ if it is not constant. The theta term is not completely trivial, despite being sort-of a total derivative, because $CS(A)$ is not gauge invariant. The Chern-Simons term is topological, and has many interesting connections with physics.

Suppose that we choose a gauge with $A^0 = 0$. Then $\partial_\mu K^\mu = \partial_0 K^0$ with $\int d^3\vec{x} K^0 = W[\vec{A}] = \frac{1}{8\pi^2}\int d^3x \epsilon^{ijk}\mathrm{Tr}(A_i\partial_j A_k - \frac{2i}{3}A_i A_j A_k)$, which is the 3d Chern-Simons action. The $A_0 = 0$ gauge fixing condition still allows gauge transformations by time independent $U(\vec{x})$. If we impose $U(|x| \to \infty) \to 1$, we can essentially replace the space $\mathbf{R^3} \to S^3$ and the remaining gauge transformations are associated with topologically non-trivial maps $S^3 \to G$. The CS action $W[A]$ is actually almost gauge invariant, but it picks up an additive integer shift under large gauge transformations, associated with the homotopy $\pi_3(G)$ of $S^3_\infty \to G$. Under a gauge transformation $U(x)$, $\vec{A} \to U(\vec{A} + i\vec{\nabla})U^{-1}$, get $W[A] \to W[A] + \frac{1}{4\pi^2}\int d^3x\epsilon^{ijk}(i\partial_j(\mathrm{Tr}(\partial_i U U^{-1}A_k))) - n(U)$, where the total derivative term can be dropped (unless we consider the case where space has a boundary) and $n[U] = \int \frac{d^3\vec{x}}{24\pi^2}\epsilon^{ijk}\mathrm{Tr}((U^{-1}\partial_i U)(U^{-1}\partial_j U)(U^{-1}\partial_k U)) \in \mathbf{Z}$ measures the topological winding. Imposing $U \to 1$ at spatial infinity (otherwise $U$ should be interpreted as a global part of the gauge group) means that $U$ maps $S^3 \to G$, and $n[U]$ measures the $\pi_3(U)$ winding number. All simple non-Abelian groups have $\pi_3(G) = \mathbf{Z}$, which can be understood as coming from $SU(2) \subset G$ and $SU(2) \cong S^3$ (to see this note that $U \in SU(2)$ can be written as $\begin{pmatrix} z_1 & -z_2 \\ z_2^* & z_1^* \end{pmatrix}$ with $(z_1, z_2)$ complex and $|z_1|^2 + |z_2|^2 = 1$; this $S^3$ can be thought of as the Euler angles in the case of the rotation group) and $\pi_3(S^3) = \mathbf{Z}$. For example, take $U \in SU(2)$ with $U = \exp(i(\frac{1}{2}\sigma^a)\alpha^a(x))$ with $\alpha^a = \hat{r}^a f(r)$ where $f(0) = 0$ and $f(\infty) = 4\pi n$: for $n = 1$ this directly maps the physical space $S^3$ to the group target space $S^3$ and has $n[U] = 1$, and more generally it has $n[U] = n$.

The fact that $W[A]$ can shift by an integer under large gauge transformations means that $e^{2\pi i K W[A]}$ is gauge invariant if $K \in \mathbf{Z}$, so a Chern-Simons term can be added to the 3d action, with quantized coefficient, without spoiling gauge invariance of the path integral.

- Consider $\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}F^{\mu\nu}F_{\mu\nu} + \frac{\theta}{16\pi^2}\mathrm{Tr}^*F^{\mu\nu}F_{\mu\nu}$ in $A_0^a = 0$ gauge:

$$\mathcal{L} = \frac{1}{g^2}\mathrm{Tr}(\dot{\vec{A}}^2 - \vec{B}^2) + \frac{\theta}{4\pi}\mathrm{Tr}\dot{\vec{A}} \cdot \vec{B},$$

so $\vec{\Pi} = \frac{\partial\mathcal{L}}{\partial\dot{\vec{A}}} = g^{-2}\vec{E} + \frac{\theta}{8\pi^2}\vec{B}$ and $\mathcal{H} = g^{-2}\mathrm{Tr}(\vec{E}^2 + \vec{B}^2) = g^2\mathrm{Tr}(\vec{\Pi} - \frac{\theta}{8\pi^2}\vec{B})^2 + g^{-2}\mathrm{Tr}\vec{B}^2$. Even though we take $A_0 = 0$, we still need to impose the EOM for $A_0$, which gives a non-Abelian version of Gauss' law, $\mathcal{D}_i E_i = 0$. As usual In QFT we do not impose EOMs as operator equations, but just that it holds for on-shell physical states, so $\mathcal{D}_i E_i|\mathrm{physical}\rangle = 0$. The Hilbert space includes non-gauge invariant states, and Gauss' law is equivalent to the condition that the physical states should be charge neutral.