215c, 4/13/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 3 reading: Tong chapter 2.**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• Continue from last time with non-Abelian gauge theory: $\psi \to U\psi$ takes $D_\mu\psi \to D_\mu^U U\psi = UD_\mu\psi$. So $D_\mu = \partial_\mu - iA_\mu$ with $A_\mu^U = U(i\partial_\mu + A_\mu)U^{-1}$.

$$F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = F_{\mu\nu} = F_{\mu\nu}^a T^a,$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc}A_\mu^b A_\nu^c.$$

• In the differential form notation, $A = A_\mu^a T^a dx^\mu$, and $F = \frac{1}{2}F_{\mu\nu}^a T_{adj}^a dx^\mu \wedge dx^\nu$ is given by $F = dA - iA \wedge A$. The i's are annoying enough that it is common to replace the Hermitian generators T_{here}^a with anti-Hermitian generators $T_{new}^a = -iT_{here}^a$ and then $A_{new} = -iA_{here}$, and $F_{new} = -iF_{here}$ so $F_{new} = dA_{new} + A_{new} \wedge A_{new}$. I'll stick with Hermitian generators.

• $F_{\mu\nu}$ is in the adjoint rep: $F_{\mu\nu} \to F_{\mu\nu}^U = UF_{\mu\nu}U^{-1}$. For $U = \exp(i\alpha^a T^a)$ and α^a infinitesimal, get $\delta F_{\mu\nu} = i[\alpha, F_{\mu\nu}]$. $F_{\mu\nu} = 0$ if and only if A_μ is pure-gauge, $A_\mu = iU\partial_\mu U^{-1}$.

The fact that $F_{\mu\nu}$ is in the adjoint representation is a key physical difference between Abelian (commuting) vs non-Abelian gauge theories. The gauge fields are always in the adjoint representation, and for Abelian ($u(1)$ and products) gauge theories the adjoint is charge neutral, and the photons are thus gauge invariant states. For non-Abelian gauge theories, the adjoint gauge fields, e.g. gluons, are not gauge invariant states. Note also that $\mathrm{Tr}F_{\mu\nu} = 0$ since $\mathrm{Tr}T^a = 0$ for simple (without added $U(1)$ factors) non-Abelian groups (e.g. $SU(N)$ vs $U(N)$, $\det U = \exp(i\mathrm{Tr}\alpha^a T^a) = 1$ gives $\mathrm{Tr}T^a = 0$. One can form gauge invariant composites, e.g. glueballs, from 2 or more gluons, e.g. $\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$.

• Recall that in $u(1)$ gauge theory, writing $F_{\mu\nu}$ in terms of A_μ ensures that the two Maxwell equations expressing absence of magnetic monopoles and currents are automatically satisfied: we can write this in relativistic notation as $\partial_\mu \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = 0$, or equivalently the Bianchi identity $\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho} = 0$. In the non-Abelian version, writing $F_{\mu\nu}$ in terms of A_μ ensures a gauge covariant version of the Bianchi identity $D_\mu F_{\rho\sigma} + D_\rho F_{\sigma\mu} + D_\sigma F_{\mu\rho} = 0$, where $(D_\mu F_{\rho\sigma}) = \partial_\mu F_{\rho\sigma} - i[A_\mu, F_{\rho\sigma}]$, i.e. $(D_\mu F_{\rho\sigma})^a = \partial_\mu F_{\rho\sigma}^a + f^{abc}A_\mu^b F_{\rho\sigma}^c$. The covariant derivatives in the Bianchi identity ensure that it holds in any gauge.

• The Lagrangian density must of course be gauge invariant, and the gauge kinetic terms come from the quadratic casimir (squaring and taking the trace): $\mathcal{L} \supset$

$-\frac{1}{2g^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$. E.g. for $G = SU(2)$, we can use a notation inspired by the rotation group, where the $j = 1$ adjoint is denoted by a 3d vector, so $\vec{F} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + \vec{A}_\mu \times \vec{A}_\nu$, and the gauge kinetic terms are $-\frac{1}{4g^2}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu}$.

Expanding it out, with $\mathrm{Tr}T^aT^b = \frac{1}{2}\delta_{ab}$, get

$$\mathcal{L}_{gauge} = -\frac{1}{4g^2}[(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + 4f^{abc}A^b_\mu A^c_\nu \partial^\mu A^{\nu a} + f^{abc}f^{afg}A^b_\mu A^c_\nu A^{\mu f}A^{\nu g}].$$

The cubic and quartic terms in A^a_μ mean that even the pure Yang-Mills case (no matter) is interacting, with non-linear EOM. The non-linearity is because the $F^{\mu\nu}$ are charged, in the adjoint representation.

- The coupling constant g is the analog of the electric coupling e, and we can try perturbation theory in $g^2/4\pi$, the analog of the fine structure constant. To do this, it is convenient to rescale $A^\mu/g \equiv \hat{A}^\mu$, so the $\hat{A}^\mu$ have canonical kinetic terms. With that normalization,

$$\mathcal{L} \supset -\frac{1}{4}(\partial_\mu \hat{A}^a_\nu - \partial_\nu \hat{A}^a_\mu)^2 - gf^{abc}\hat{A}^b_\mu \hat{A}^c_\nu \partial^\mu \hat{A}^{\nu a} - \frac{1}{4}g^2 f^{abc}f^{afg}\hat{A}^b_\mu \hat{A}^c_\nu \hat{A}^{\mu f}\hat{A}^{\nu g}.$$

The last two terms mean that there are Feynman interaction vertices with three gauge fields $\sim gf^{abc}$, and with four gauge fields $\sim g^2 f^{abc}f^{afg}$. We will discuss the Feynman rules in detail soon – this is just an appetizer.

- Gauge theories can also have matter fields, in various representations. For a $u(1)$ gauge theory we could have just the photon, which is called pure Maxwell theory and is a free theory, or we could have e.g. QED with the photon and an electron matter field. For a non-Abelian gauge theory we can have just the gauge fields, which is called pure Yang-Mills theory, and it is a strongly coupled theory in the IR; as we will discuss, it is asymptotically free in the UV, and RG flows to strong coupling in the IR. We can also add matter, e.g. the quark fields in the case of QCD. Suppose that we have a Fermion ψ in a representation R of the gauge group. To be concrete, we can take $su(N)$, with a Dirac Fermion in the fundamental, with

$$\mathcal{L} = -\frac{1}{4g^2}F^a_{\mu\nu}F^{a,\mu\nu} + \bar{\psi}(i\not{D} - m)\psi.$$

Write out explicitly the suppressed color indices ψ^i, $\bar{\psi}_j$, $m\delta^j_i$, $i\delta_{ij}\not{\partial} + \not{A}^a T^{a,j}_i$. If the $su(N)$ were a global symmetry, Noether's theorem would give the conserved matter currents $j^{a,\mu} = \bar{\psi}_j\gamma^\mu T^{a,j}_i\psi^i$. Note that this current is in the adjoint representation, and $\mathcal{L} \supset$

$A^a_\mu j^{\mu,a}$. Because of the gauge field terms involving f^{abc}, it can be checked that this current is no longer conserved, $\partial_\mu j^{a,\mu} \neq 0$, but is instead covariantly conserved $D_\mu j^{a,\mu} = 0$. This is expected and general: derivatives should be replaced with appropriate covariant derivatives. Indeed, under a gauge transformation $\psi \to U\psi$, writing $j^\mu = j^{a,\mu} T^a_{Ad}$, we have $j^\mu \to U j^\mu U^{-1}$, so $\partial_\mu j^\mu$ does not transform nicely and setting it to zero would not be gauge invariant; writing D_μ fixes this. There is no conserved, gauge invariant current – only a covariantly conserved one. This means that there is no associated conserved charge.

• Consider the EL equations for A^a_μ with $\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{matter}$ where $\mathcal{L}_{matter}$ could be e.g. the example above. One way to write the EL equations is as $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a_\nu)} = \frac{\partial \mathcal{L}}{\partial A^a_\nu}$. The LHS is $-g^{-2}\partial_\mu F^{a,\mu\nu}$ (assuming that $\mathcal{L}_{matter}$ does not have any $\partial_\mu A^a_\nu$ terms, as above). The RHS is $j^{\nu a} + g^{-2} f^{abc} A^b_\mu F^{c,\nu\mu}$. Acting with ∂_ν gives zero on both sides, but the RHS is not a good current choice, because it is not gauge covariant - it is a mess because of the additive shift in A^b_μ. Again, it is better to rewrite everything with covariant derivatives. Then get $D_\mu \frac{\partial \mathcal{L}}{\partial(D_\mu A^a_\nu)} = \frac{\partial \mathcal{L}_{matter}}{\partial A^a_\nu} \equiv j^a_\nu$, and both sides transform in the adjoint representation. The LHS is $-g^{-2} D_\mu F^{\mu\nu}$, and it can be checked that D_ν gives zero on both sides.

• As in E&M, we can consider how a wavefunction or the path integral is affected if we move a charged object along some path. The charged object could either be a quanta of one of the dynamical fields, or an external charged source. As in E&M it can be either electrically charged or magnetically charged, or dyonically charged. For an electrically charged object in E&M, we get e.g. $\psi \sim e^{i\int A_\mu j^\mu/\hbar} \sim e^{iq_e \int A_\mu dx^\mu/\hbar}$. In the non-Abelian case, we can do something similar, where we replace q_e with T^a_R for the group representation of the object. Because the A^a_μ do not commute, we need to be careful with the ordering. As we saw in 215a, the path integral automatically gives time ordering, so the correct object to consider is $OWL[x_i, x_f; C]_R = \mathcal{P}\exp(i\int dx^\mu A^a_\mu T^a_R)$, where $\mathcal{P}$ is path time ordering the operators along the space curve C (which has endpoints x_i and x_f), and OWL stands for open Wilson line. Under a gauge transformation, $OWL[x_i, x_f; C]_R \to U(x_i)_R OWL[x_i, x_f; C]_R U^\dagger(x_f)_R$. The subscript is a reminder that everything is in a representation R, but it gets tedious to keep writing it so it is usually left implicit. To get something gauge invariant, we can consider the Wilson loop, where C is a closed curve: $W[C]_R = \mathrm{Tr}_R \mathcal{P}(\exp i \oint A/\hbar)$. If we consider instead a magnetically charged object, the analogous object is called the 't Hooft loop. I will discuss these quantities more later.