215c, 4/6/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 2 reading: Tong chapter 1, and start chapter 2.**
http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• Continue from last time, with Dirac quantization. Rather than having a Dirac string, we can achieve $\nabla \cdot \vec{B} \neq 0$, and still have $\vec{B} = \nabla \times \vec{A}$, via having $\vec{A}$ defined only in patches. This will also be a warmup for the 't Hooft Polyakov monopole, to be discussed later. Consider $A^N_\phi = \frac{q_m}{4\pi r}\frac{(1-\cos\theta)}{\sin\theta}$, which is singular at $\theta = \pi$, and $A^S_\phi = -\frac{q_m}{4\pi r}\frac{(1+\cos\theta)}{\sin\theta}$, which is singular at $\theta = 0$; both lead to $\vec{B} = q_m \hat{r}/4\pi r^2$, corresponding to a charge q_m magnetic monopole at the origin, $\int_{S^2} \vec{B}\cdot d\vec{a} = \int_{S^2} F = q_m$.

We can try to get an everywhere well defined gauge field by taking $A_\mu = A^N_\mu$ in the $0 \leq \theta < \pi/2$ Northern hemisphere, and $A_\mu = A^S_\mu$ in the $\pi \geq \theta > \pi/2$ Southern hemisphere. They disagree at $\theta = \pi/2$, where they're patched together, but that's OK because the difference is a gauge transformation: $A^N_\phi - A^S_\phi = \frac{1}{r\sin\theta}\partial_\phi \alpha = \nabla_\phi \alpha$ with $\alpha = q_m\phi/2\pi$. The only caveat is that the wavefunction must be single valued and since $\psi \to e^{-iq_e\alpha/\hbar}\psi$ under a gauge transformation, the wavefunction in the patches differ by a factor of $e^{-iq_e q_m \phi/2\pi\hbar}$. This must be periodic under $\phi \cong \phi + 2\pi$, i.e. $\alpha \cong \alpha + q_m$. This gives another way to see Dirac quantization: under a gauge transformation $\psi \to e^{-iq_e\alpha/\hbar}\psi$ and $\alpha \cong \alpha + q_m$ does nothing if $q_e q_m \in 2\pi\hbar\mathbf{Z}$.

• Summary: compact $u(1) \leftrightarrow$ gauge parameter α lives on a circle, $\alpha \cong \alpha + q_m$. Allows for magnetic monopoles and implies that electric charge is quantized. So when I discuss $u(1)$ gauge theory, I usually assume that it is compact and thus that charges are quantized. It fits with observation, and it is required in grand unification scenarios where the $u(1)_Y$ gauge group of the SM is embedded in a larger compact group, like $su(5)$ or $so(10)$. If the charge is quantized, we can rescale A_μ such that the basic quanta of charge has $q_e = 1$, taking $q_m = 2\pi\hbar$ for the minimum magnetic monopole. We usually set $\hbar = 1$, but it's nice to note that the quantization is a quantum effect, quantized in units of $\hbar$.

• Since $L \supset -q_e\phi + \frac{q}{c}\vec{v}\cdot\vec{A}$, $\vec{p} = \gamma m\vec{v} + \frac{q}{c}\vec{A}$. The conserved angular momentum of a charge q_e in the background of a monopole is $\vec{L} = \vec{r}\times\vec{p} - \frac{q_e q_m}{4\pi}\hat{r}$. Then $\vec{L}\cdot\hat{r} = -\frac{q_e q_m}{4\pi}$ and quantization of angular momentum $L_z \in \frac{1}{2}\hbar\mathbf{Z}$ gives Dirac quantization again.

• The 2-form $F/2\pi$ is called $c_1(F)$, the first Chern class, where the name comes from mathematics. In general, $c_n(F)$ comes from the $F^\wedge n$ term in Taylor expanding $e^{F/2\pi}$ with wedge products. The second Chern class is thus $c_2(F) = \frac{1}{8\pi^2}F\wedge F$. The normalizations are nice in terms of quantization conditions. For example, for the magnetic monopole at the origin $\int_S F = q_m$ and if we pick $q_e = 1$ then Dirac quantization gives $\int_S (F/2\pi\hbar) \in \mathbf{Z}$.

• It is illuminating to consider QFTs on a compact spacetime, e.g. suppose that x^1 is a circle, $x^1 \sim x^1 + 2\pi R_1$. Under $x^1 \to x^1 + 2\pi R_1$, $\psi \to e^{-iq_e \oint A_1 dx^1/\hbar}\psi$. Note that $A_1 \sim A_1 + \hbar/q_e R_1$, the gauge field becomes periodic with radius inverse to the radius of space. Can consider $A_1 \to A_1 + \partial_1 \alpha$ with $\alpha = x^1\hbar/q_e R_1$: it winds the circle in space around the circle in gauge transformations with non-trivial $\pi_1(S^1)$. So $A_1 \to A_1 + \hbar/q_e R$ are related by a (large) gauge transformation, and thus physically equivalent.

Suppose now that x^2 is also a circle, $x^2 \sim x^2 + 2\pi R_2$, and $A_2 \sim A_2 + \hbar/R_2$. Now we can take e.g. $A_2 = \hbar n x^1/2\pi q_e R_1 R_2$, so as we wind once around the x^1 cycle of the space torus we wind $n \in \mathbf{Z}$ times around the A_2 cycle of the gauge field's torus. This leads to $B_3 = \hbar/2\pi q_e R_1 R_2$ and thus magnetic flux through the torus $\int_{T^2} F = 2\pi n\hbar/q_e$. Setting $q_e = 1$ and $\hbar = 1$, this is a configuration with $\int c_1(F) = n$.

• Recall dimensional analysis, use Δ to denote operator's mass dimension, so $\Delta(S) = 0$, $\Delta(\mathcal{L}) = 4$, $\Delta(D_\mu) = \Delta(\partial_\mu) = \Delta(A_\mu) = 1$. $\Delta(F_{\mu\nu}) = 2$. The gauge kinetic term is $\mathcal{L} \supset -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}$ so $\Delta(e^2) = 0$; a non-zero beta function is a quantum correction to this.

• There is another gauge and Lorentz invariant operator with $\Delta = 4$, i.e.

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 8\vec{E}\cdot\vec{B} = 2\partial_\mu(\epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}).$$

We can add it to $\mathcal{L}$ with coefficient, in a normalization to be made precise below, called θ, which is a dimensionless (even including quantum loops) coupling constant. Superficially, such a term is a total derivative, but it's not a total derivative of something gauge invariant. In the form notation, where $F = dA$ is a 2-form, we are discussing the 4-form $F \wedge F = d(A \wedge dA)$. It looks exact, and then Gauss' law gives $\int_{M_4} F \wedge F = \int_{\partial M_4} A \wedge dA$.

But that is not the whole story. Just as a magnetic monopole can have $\int_S F = q_m \neq 0$, with S a closed surface $\partial S = 0$, despite seeming to have $F = dA$, likewise we can have $\int_{M_4} F \wedge F \neq 0$ even if $\partial M_4 = 0$. As with a magnetic monopole, the integral is a quantized, topological invariant, associated with non-trivial winding of the gauge field. For the Abelian $U(1)$ case, it is a product of monopole numbers, whereas for the non-Abelian case it is an independent invariant (the second Chern class vs first Chern class squared).

To illustrate it, suppose that take spacetime M_4 to be a Euclidean T^4, with all $x^\mu \sim x^\mu + 2\pi/R_\mu$ (this has $\partial M_4 = 0$). Then we can do an analogous construction for the electric field E_3 via e.g. $A_3 = \hbar n' x^0/2\pi q_e R_0 R_3$ and then $\int_{T_2'} dx^0 dx^3 E_3 = 2\pi\hbar c n'/q_e$. Combining the $\vec{E}$ and $\vec{B}$, this configuration has $\int_{T^4} \vec{E}\cdot\vec{B} = N(4\pi^2\hbar^2 c/q_e^2)$ with $k = nn' \in \mathbf{Z}$. k is called the instanton number. It is the integrated second Chern class. Instantons require a lot more discussion, and they generally do not play much of a role in $u(1)$ gauge theory (modulo UV modifications). We will discuss them much more soon, in the context of non-Abelian gauge theories, where they are associated with non-trivial maps of $S^3_\infty \to S^3 \cong SU(2) \subset G$. The θ term's role in the $u(1)$ case is primarily if there is a boundary.

• Now consider adding the term $\sim \vec{E}\cdot\vec{B}$ to $\mathcal{L}$, taking care with the normalization:

$$S_{\theta,u(1)} = \frac{\theta}{4\pi^2\hbar c}\int d^4x \vec{E}\cdot\vec{B} = \frac{\theta}{32\pi^2\hbar c}\int d^4x \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \theta\int \tfrac{1}{2}c_1(F)\wedge c_1(F).$$

In terms of forms, $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = 2\pi c_1(F)$ and $F\wedge F = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} * d^4x$ (e.g. can take $F = B_3 dx_1\wedge dx_2 + E_3 dx_3 \wedge dt$ and then $F\wedge F = 2\vec{E}\cdot\vec{B}dx_1\wedge dx_2\wedge dx_3\wedge dt$). When we discuss the non-Abelian case, we will instead take

$$S_{\theta,non-Abelian} \supset \theta\int c_2(F), \qquad c_2(F) = \tfrac{1}{2}\mathrm{Tr}\frac{F}{2\pi}\wedge\frac{F}{2\pi}.$$

It turns out that, in both cases, S_θ can only take quantized values, and the normalization is chosen to make the quantization condition nice: $S_\theta \in \hbar\theta k$ with $k\in\mathbf{Z}$ an integer (called the instanton number). Then $e^{iS/\hbar}$ in the path integral is periodic in $\theta \cong \theta + 2\pi$.

θ is a parameter (coupling constant) called the *theta angle*, since it is a 2π periodic. Since P and T sent $\theta\to-\theta$, the theta parameter violates P and T separately, but preserves PT and CPT (of course, since the theory is Lorentz invariant). Note that both $\theta = 0$ and $\theta = \pi$ preserve P and T; there are some interesting aspects to these two values.