215c, 4/1/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 1 reading: Tong chapter 1, and start chapter 2.**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• Caution about conventions: for QED, my notation last time was $D_\mu = \partial_\mu + iqA_\mu$

$$\mathcal{L} \supset -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \supset -A_\mu J^\mu, \qquad \text{with} \quad J^\mu = q\bar{\psi}\gamma^\mu\psi.$$

The gauge fields do not have canonical kinetic term. To make it canonical, we take $A_\mu = e\hat{A}_\mu$ and then $D_\mu = \partial_\mu + iqe\hat{A}_\mu$. I will discuss charge quantization today, and then e.g. $q \in \mathbf{Z}$ whereas e could be the charge of the electron. The EOM is $\partial_\mu \hat{F}^{\mu\nu} = \hat{J}^\mu$, where $\hat{F}^{\mu\nu} = \partial^\mu \hat{A}^\nu - \partial^\nu \hat{A}^\mu$ and $\hat{J}^\mu = eJ^\mu = eq\bar{\psi}\gamma^\mu\psi$; these are fairly standard conventions. But the gauge transformation has a minus sign: it's $\psi \to e^{-iq\alpha(x)}\psi$ with $A_\mu \to A_\mu + \partial_\mu\alpha$.

An alternative notation is to take $A_\mu^{new} = -A_\mu^{old}$. Then the gauge transformation is $\psi \to e^{+iq\alpha(x)}\psi$ with $A_\mu^{new} \to A_\mu^{new} + \partial_\mu\alpha$, which looks nice, and $D_\mu\psi = (\partial_\mu - iqA_\mu^{new})\psi$. Then $F^{\mu\nu,new} = -F^{\mu\nu}$ and we can either redefine J^μ with a minus sign to get Maxwell's equations to look the same, or leave J^μ alone and end up with $\mathcal{L} \supset +A_\mu^{new}J^\mu$. The latter notation is often used in the context of non-Abelian gauge fields.

Another notational issue in the non-Abelian case, which I mentioned last time, is whether to take $[T^a, T^b] = if^{abc}T^c$, with the familiar i from the angular momentum commutation relations, or to take $T^a = iT^{a,new}$ and then $[T^{a,new}, T^{b,new}] = f^{abc}T^{c,new}$. Then the generators are e.g. anti-Hermitian, rather than Hermitian, for $SU(N)$. I will mostly use the Hermitian generator notation.

An object ψ in the fundamental rep transforms as $\psi \to U\psi$, and an object $\mathcal{O}$ in the adjoint rep transforms as $\mathcal{O} \to U\mathcal{O}U^{-1}$. In the Lie algebra, the adjoint is represented by $(T^a)^{bc} = -if^{abc}$ (in the notation where T^a are Hermitian). E.g. for $SU(2)$ with $f^{abc} = \epsilon^{abc}$ this leads to the standard $j = 1$ matrix elements of $J^a = \hbar T^a$. (HW to check these.)

• This seems like a good time to mention the connection (also in the technical sense) with differential forms. The following is just a brief sketch, to give a flavor of how it applies in the case of gauge theories. It is useful to occasionally use this language and notation.

A function e.g. $\alpha(x)$ is called a 0-form. The differential operator d takes p forms to $p+1$ forms, e.g. $d\alpha = \partial_\mu\alpha dx^\mu$ is a 1-form. One forms that can be written as d of a zero form are called *exact*, so $d\alpha$ is an example of an exact 1-form. Forms that are annihilated by d are called *closed*. Now $d^2 \equiv \partial_\mu\partial_\nu dx^\mu \wedge dx^\nu = 0$, because the wedge product of two one-forms is odd under interchange whereas the partial derivatives commute; so all exact forms are closed. But not all closed forms are exact; closed forms modulo exact forms are called *cohomology*.

We can also write $A = A_\mu dx^\mu$ as a 1-form, and gauge invariance is the statement that physics doesn't care if we shift it by any exact 1-form. The field strength tensor can be written as a 2-form $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$, where again the wedge product of two one-forms is odd under exchange, fitting with the fact that $F_{\mu\nu} = -F_{\nu\mu}$. The Maxwell's equations that state there are no magnetic charges (as far as we know, and we will consider magnetic monopoles shortly) say that F is a closed form: $dF \equiv \partial_\lambda F_{\mu\nu} dx^\lambda \wedge dx^\mu \wedge dx^\nu = 0$. We locally solve that by writing F as an exact form: $F = dA$, and gauge invariance of F under $A \to A + d\alpha$ follows from $d^2 = 0$. In cases with magnetic flux (e.g. a solenoid, or in the vortex strings of the Abelian Higgs model that you might have met in a HW assignment last quarter), then actually $dF \neq 0$. It is still sometimes, useful to locally write $F = dA$, but with an A that has something that makes it globally ill-defined or having jumps, e.g. in polar coordinates we can write $d\phi$ as something that looks like an exact 1-form, but integrating it around a closed counter that encircles the origin gives 2π rather than 0; this happens because $d\phi$ is ill-defined at the origin.

Now $F \wedge F \sim \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} vol$ where $vol = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ is the spacetime volume 4-form. $F \wedge F$ is a topological 4-form that can be integrated over spacetime. By contrast $F_{\mu\nu} F^{\mu\nu} vol \sim F \wedge *F$ where $*$ is called the Hodge dual, which takes a p form to a $D - p$ form in D dimensions (here $D = 4$) by contracting indices with an epsilon tensor, $*F^{\mu\nu} \equiv \tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ (which differs from $F_{\mu\nu}$ by $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$). There are some hidden factors of the metric in writing $F \wedge *F$ so it is not topological.

• There are various versions of a $u(1)$ gauge theory that differ by global considerations. A basic issue is whether the $u(1)$ group manifold is really a circle (compact), or the real line. If it is a circle, the charges must be rational and we can normalize things such that they are integers. If we only see quantized electric charges (as in the real world E&M), the group can be either compact or non-compact – we need more information to distinguish the two cases. If we later find an irrational electric charge, we then know that it must be the non-compact case. If instead we find a magnetic monopole, then we know that it must be the compact case: compact → magnetic monopoles → charge quantization. If the $u(1)$ unifies into a non-Abelian group (we will discuss an $su(2)$ example soon), then we know that it must be the compact case.

As a warmup and reminder of the Aharanov-Bohm effect, suppose that there is an infinitesimally thin solenoid along the $\hat{z}$ axis, so $\vec{B} = \Phi\delta(x)\delta(y)\hat{z}$, where Φ is the magnetic flux in the solenoid. By Stoke's law, $\oint_C \vec{A}\cdot d\vec{x} = \Phi$ if C circles the solenoid, and we can take e.g. $\vec{A} = (\Phi/2\pi r)\hat{\phi} = \vec{\nabla}(\Phi\phi/2\pi)$. So $\vec{A}$ is almost pure gauge, but not quite given that ϕ is only locally defined and has a 2π jump upon encircling the $\hat{z}$ axis.

To simplify things, suppose that a particle of charge q is restricted to live on a ring of radius $r = R$, which encircles the flux Φ. The Lagrangian is $L = \frac{1}{2}mR^2\dot{\phi}^2 + \frac{\theta}{2\pi}\dot{\phi}$, where $\theta \equiv q\Phi$. The last term is superficially a total derivative, and indeed it is topological because of that – e.g. it drops out of the EL equations of motion – but it is not trivial because $\phi \sim \phi + 2\pi$. Get $p_\phi = \partial L/\partial\dot{\phi} = mR^2\dot{\phi} + \frac{\theta}{2\pi}$ and $H = \frac{1}{2mR^2}(p_\phi - \frac{\theta}{2\pi})^2$. In QM, we quantize via $p_\phi \to -i\partial_\phi$. The p_ϕ eigenstates are $\psi_n(\phi) = \langle\phi|n\rangle = \frac{1}{\sqrt{2\pi R}}e^{in\phi}$, with $p_\phi = n$. The $\psi_n(\phi)$ are energy eigenstates, with $E_n = \frac{1}{2mR^2}(n - \frac{\theta}{2\pi})^2$; note that this spectrum is invariant under $\theta \to \theta + 2\pi$. Indeed, if we try to eliminate the almost-pure-gauge $\vec{A}$ by a gauge transformation $\vec{A} \to \vec{A} + \nabla\alpha$ with $\alpha = -\Phi\phi/2\pi$ then $\psi \to \psi' = e^{i\theta\phi/2\pi}\psi$. Under $\phi \to \phi + 2\pi$, the original ψ is invariant but $\psi' \to e^{i\theta}\psi'$. This shows that θ can affect the physics only $\theta \notin 2\pi\mathbf{Z}$, and that $\theta \sim \theta + 2\pi$. The θ here will have similarities with the θ parameter in gauge theory.

Dirac understood the above effect and argued (decades before Aharanov-Bohm), that there can be magnetic monopoles, which could be imagined as being the endpoints of fictional, Dirac string solenoids, and that the pretend string will be unobservable if the corresponding $\theta \in 2\pi\mathbf{Z}$, which is Dirac's quantization condition on electric and magnetic charges. Again, consider QM for simplicity, and since the wavefunction $\psi \sim e^{iS}$ with $S \supset -\int qA_\mu dx^\mu$ moving the particle along some path C takes $\psi \to \exp(-iq\int_C A_\mu dx^\mu)\psi$. If the path is closed, $\psi \to \exp(iq\oint_C \vec{A}\cdot d\vec{x})\psi$. If there is a magnetic monopole somewhere, then $\nabla\cdot\vec{B} = q_m\delta^3(\vec{x}-\vec{x}_0)$ and we can then only locally define $\vec{A}$. Using Gauss' law, $\oint_{C=\partial S}\vec{A}\cdot d\vec{x} = \int_S \vec{B}\cdot d\vec{a}$, but there are two choices of S (e.g. for the equator we can pick the Northern or Southern hemisphere). The two choices differ by $\int_{S-S'=\partial V}\vec{B}\cdot d\vec{a} = \int_V \nabla\cdot\vec{B}dV$, so if there is a magnetic monopole inside V we get $\oint \vec{A}\cdot d\vec{x}$ is ambiguous by an additive shift of q_m. This ambiguity does not affect ψ as long as $q_e q_m \in 2\pi\hbar\mathbf{Z}$.