215c, 6/3/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 10 reading: Tong chapter 3 on anomalies.**
http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• Recap from last time: the anomaly in 4d is associated with a breaking of the symmetry in a current 3-point function, or equivalently a violation of current conservation in the presence of background fields. We considered the simplest example, associated with violation of the axial $U(1)_A$ current (associated with $\psi \to e^{i\lambda_A \gamma^5}\psi$ with Noether current $J_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$) in the presence of $U(1)_V$ gauge or background gauge currents ($\psi \to e^{i\lambda_V}\psi$, with $J^\mu = \bar{\psi}\gamma^\mu\psi$). The result was seen to be $\partial_\mu J_5^\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ (in the $A_\mu^{best} = eA_\mu^{pert}$ normalization, or multiplied by e^2 in terms of A_μ^{pert}). The anomaly is independent of the RG scale (in the normalization where the factor of e^2 is absorbed into the correctly normalized gauge fields. It follows from this that the anomaly is one-loop exact (Adler Bardeen theorem), since higher loops would depend on $e^2(\mu)$ and not be RG invariant. Since the anomaly is dimensionless, it cannot depend on any masses. It is completely robust, because it is fundamentally topological.

We obtained the result in a few different ways: (1) the physical argument based on quantizing the Landau levels in a $\vec{E}\cdot\vec{B}$ background, and accounting for how the electric field shifts the Dirac sea in a direction that correlates with L or R moving chirality; (2) the $U(1)_A$ non-invariance of the path integral measure, and hence $\det(i\not{D}e^{-(i\not{D})^2/\Lambda^2})$, where the term with Λ is to regulate the infinite product of eigenvalues and we should take $\Lambda \to \infty$ at the end; (3) point splitting the J_5^μ current to $J_5^\mu = \lim_{\epsilon\to 0}\bar{\psi}(x+\frac{1}{2}\epsilon)\gamma^\mu\gamma^5\exp(-ie\int_{x-\frac{1}{2}\epsilon}^{x+\frac{1}{2}\epsilon}A_\nu(y)dy^\nu)\psi(x-\frac{1}{2}\epsilon)$ and taking the $\epsilon\to 0$ limit appropriately. We will today discuss method (4): the original calculation of the one-loop Feynman diagram. First, a few more comments.

• As emphasized last time, the anomaly is topological and independent of any continuous parameters, in particular the coupling e, and any mass m, and it can arise only for massless Fermions. Note that a mass term $\mathcal{L} \supset -m\bar{\psi}\psi$ explicitly violates $U(1)_A$, but not $U(1)_V$, which is why the anomaly shows up in the $U(1)_A$ conservation law. The anomaly is independent of any renormalization scheme or counterterms. If we wanted to, we could add counterterms that restore $U(1)_A$ symmetry but they would then violate $U(1)_V$: the anomaly is an incompatibility in preserving both, and we choose to sacrifice $U(1)_A$ and preserve $U(1)_V$, which is in particular mandatory if we gauge $U(1)_V$.

• We saw in our discussion of the theta angle that $\frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \partial_\mu K^\mu$, but K^μ is not gauge invariant so the RHS of the anomaly equation is not exactly a total derivative. If it were, we could simply redefine the current $J_5^\mu \to J_5^\mu - K^\mu$ to get something conserved. This sort-of works in perturbation theory, but not non-perturbatively: $\int d^4x \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ can be non-zero, recall the instanton winding number. So even though the anomaly can be seen from a one-loop calculation, the actual charge violation is non-perturbative.

• Let's emphasize again the connection between the anomaly and shifting the theta angle. Also, the sign can be a pain to get right (e.g. active vs passive view of a transformation), so let's try to be extra careful. The $U(1)_A$ transformation is $\psi \to e^{i\lambda_A\gamma^5}\psi$. The path integration measure $[d\psi][d\bar{\psi}]$ picks up Jacobian $J = \exp(-i\mathrm{Tr}\lambda_A\gamma^5 e^{-(i\not{D})^2/\Lambda^2})$, where the minus sign in the exponent is because Grassmann integrals pick up an inverse Jacobian determinant vs regular integrals (recall the related minus sign for each Fermion loop). The result is $J = \exp(-\frac{i}{16\pi^2}\int d^4x\lambda_A(x)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma})$, which has the same effect as shifting the θ parameter in the path integral $\exp(\frac{i}{32\pi^2}\int d^4x\theta\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma})$ by $\theta \to \theta - 2\lambda_A$. The 2 is because we could have rotated the chiral parts separately, $\psi_\alpha \to e^{i\lambda_\psi}\psi_\alpha$ and $\tilde{\psi}_\alpha \to e^{i\lambda_{\tilde{\psi}}}\tilde{\psi}_\alpha$, and then $\theta \to \theta - \lambda_\psi - \lambda_{\tilde{\psi}}$. If we rotate the Fermion by $\psi \to e^{i\lambda_A\gamma^5}\psi$ and also shift $\theta \to \theta + 2\lambda_A$, then the theory would be invariant. The broken symmetry can be regarded as restored if we could assign $U(1)_A$ charge $+2$ to $e^{-S_{inst}+i\theta}$.

• Here is a bit more about the Atiyah-Singer index of the Dirac operator. If $i\not{D}\psi_n = \lambda_n\psi_n$, then $\gamma^5\psi_n$ is also an eigenvector, with eigenvalue $-\lambda_n$. Thus ψ_n and $\gamma^5\psi_n$ are orthogonal functions if $\lambda_n \neq 0$, so ψ_n cannot be an eigenstate of γ^5 unless $\lambda_n = 0$. If we choose to diagonalize γ^5 on the space of spinors, then $\gamma^5 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $i\not{D} \to \begin{pmatrix} 0 & i\not{D}_- \\ i\not{D}_+ & 0 \end{pmatrix}$, where $\not{D}_+$ acts on positive chirality spinors to give negative chirality, and $i\not{D}_- = (iD_+)^\dagger$ does the opposite. The index is defined to be $Ind(i\not{D}) = n_+ - n_-$ where n_+ is the number of positive chirality solutions with $\lambda_n = 0$, and n_- is the number of negative chirality solutions. This can be written as $\mathrm{Tr}\gamma_5 \exp^{-(i\not{D})^2/\Lambda^2}$ and the contributions from non-zero eigenvalues cancel in the trace. The index of the Dirac operator is $Index(i\not{D}) = n_+ - n_- = \int d^4x\partial_\mu J_A^\mu = \frac{1}{16\pi^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 2k$.

Here k is the instanton number, which is an integer $k \in Z$. For $U(1)$, we mentioned that that there are not finite action instanton configurations, but when we replace $U(1)_V$ with a non-Abelian group G, e.g. $SU(N_c)$, then k measures the $\Pi_3(G)$ winding number.

The anomaly is the statement that the instanton has chiral Fermion zero-modes. Recall that in Euclidean spacetime the Lorentz group becomes $SU(2)_L \times SU(2)_R$ and the instanton winds say $SU(2)_L$ around in the group G (via the 't Hooft matrices), and the anti-instanton winds say $SU(2)_R$. So it is not shocking that the instanton can break $L \leftrightarrow R$ with zero modes carrying net chirality, and the anti-instanton has opposite chirality.

• Zero modes of Fermionic operators lead to zero in the path integral, by the rules of Grassmann integration: $\int d\psi_0 1 = 0$, so if ψ_0 drops out of the action, because it is a zero mode of $i\not{D}$, the instanton configuration only contributes if some operator soaks up the zero modes. This is called the 't Hooft operator – it is a product of Fermions, corresponding to the zero modes in an instanton configuration, and weighted by $e^{-S_{inst}+i\theta}$. The presence of a ψ_α and $\tilde{\psi}_\alpha$ Fermi zero mode means that $U(1)_A$ is broken to Z_2 by instantons, since an instanton can lead to a chiral condensate $\langle \tilde{\psi}_\alpha \psi_\alpha \rangle = e^{-S_{inst}+i\theta}$. Note that the LHS picks up a phase $e^{2i\lambda_A}$ under the chiral rotation, and the broken symmetry would be restored if we shifted $e^{-S_{inst}+i\theta}$ in the 't Hooft vertex by that same phase.

• The anomaly can be expressed in terms of the current 3-point function $\Gamma^{\mu\nu\rho}(x_1, x_2; x_3) = \langle J^\mu_V(x_1) J^\nu_V(x_2) J^\rho_A(x_3) \rangle$, which we Fourier transformed to

$$\int d^4x_1 d^4x_2 d^4x_3 e^{ip_1x_1} e^{ip_2x_2} e^{ikx_3} \Gamma^{\mu\nu\rho}(x_1, x_2; x_3) = \Gamma^{\mu\nu\rho}(p_1, p_2; k)(2\pi)^4 \delta^4(p_1 + p_2 + k).$$

It should be understood that $k = -(p_1 + p_2)$ in what follows. Note that $\Gamma^{\mu\nu\rho}(p_1, p_2; k) = \frac{\delta}{\delta A_\mu(p_1)} \frac{\delta}{\delta A_\nu(p_2)} \langle J^\rho(k) \rangle |_{A=0}$ and the anomaly is the statement that $k_\rho \Gamma^{\mu\nu\rho}(p_1, p_2; k) \neq 0$. The one-loop contribution to the 3-point function is $\Gamma^{\mu\nu\rho}(p_1, p_2; k) = F^{\mu\nu\rho}(p_1, p_2; k) + F^{\nu\mu\rho}(p_2, p_1; k)$ where the two diagrams differ by exchange of the photons and

$$F^{\mu\nu\rho}(p_1, p_2; k) = i \int \frac{d^4q}{(2\pi)^4} \mathrm{Tr}\left(\gamma^5 \gamma^\rho (\not{q} + \not{k})^{-1} \gamma^\mu (\not{q} - \not{p}_2)^{-1} \gamma^\nu \not{q}^{-1}\right)$$

(the i is from the Euclidean d^4q). The general form consistent with the γ^5 and also Bose symmetry in the two photons is

$$\Gamma^{\mu\nu\rho}(p_1, p_2) = A(p_1, p_2)\epsilon^{\mu\nu\rho\sigma}(p_1 - p_2)_\sigma + (B(p_1, p_2)(p_1 - p_2)^\mu \epsilon^{\nu\rho\sigma\kappa} + (exch))p_{1\sigma} p_{2\kappa}.$$

We can add counterterms $\mathcal{L}_{ct} = c_1 \epsilon^{\mu\nu\rho\sigma} A^5_\mu A_\nu \partial_\rho A_\sigma$, with coefficient such that J^μ and J^ν are conserved. Power counting shows that A can have a linear divergence (an apparent quadratically divergent term vanishes because $\epsilon^{\mu\nu\rho\sigma} q^\rho q^\sigma = 0$, while B is finite. We require J^μ_V conservation, so $p_{1\mu}\Gamma^{\mu\nu\rho} = p_{2\nu}\Gamma^{\mu\nu\rho} = 0$, which determines A in terms of

B. Now consider $ik_\rho \Gamma^{\mu\nu\rho}$, corresponding to $\partial_\rho J_A^\rho$ in the correlation function. The two terms, from the two diagrams, superficially cancel, if we could shift the momentum integration variable. The result has the form $ik^\rho \Gamma^{\mu\nu\rho} = \int \frac{d^4q}{(2\pi)^4}(F^{\mu\nu}(q+a) - F^{\mu\nu}(q))$ where $a^\mu = p_1^\mu - p_2^\mu$. Upon Taylor expanding the integrand in a^μ, accounting for the degree of divergence, and using $a^\rho \int \frac{d^4q}{(2\pi)^4}\frac{\partial}{\partial q^\rho}F^{\mu\nu} \to \lim_{q\to\infty}\frac{\Omega_4}{(2\pi)^4}q^2(a\cdot q)F^{\mu\nu}(q)$. The upshot is $ik_\rho \Gamma^{\mu\nu\rho} = \frac{e^2}{2\pi^2}\epsilon^{\mu\nu\rho\sigma}p_{1,\rho}p_{2,\sigma}$. This re-derives the anomaly $\partial_\mu J_A^\mu = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$.

The anomaly is physical, and regulator and scheme independent. Another regulator choice is Pauli Villars. Let $\Gamma^{\mu\nu\rho}(p_1, p_2; k)_m$ denote the quantity where the massless Fermion propagators are replaced with that of a Fermion of mass m; this explicitly violates $U(1)_A$, but we will take $m \to 0$ in the end. The PV method is to consider $\lim_{m\to 0}\lim_{M\to\infty}\Gamma^{\mu\nu\rho}(p_1, p_2; k)_m - \Gamma^{\mu\nu\rho}(p_1, p_2; k)_M$, and the divergent parts of the integrals cancel and the finite part is as above.

• Consider a more general Abalian theory, with matter chiral fermions ψ_i of charge q_i^G under some $U(1)_G$ symmetry that generalizes $U(1)_V$. If the spectrum of charges is invariant under $q_i^G \to -q_i^G$, where for every ψ_i of charge q_i^G there is a paired $\tilde{\psi}_i$ of opposite charge, then $U(1)_G$ is said to be vector-like and the chiral Fermions ψ_i and $\tilde{\psi}_i$ can be re-packaged as Dirac Fermions. In that case, $U(1)_G$ is compatible with mass terms $\mathcal{L} \supset -m\psi_{i\alpha}\tilde{\psi}_{i\beta}\epsilon^{\alpha\beta} + h.c.$. Vector-like symmetries can never have anomalies, and we will not assume $U(1)_G$ is vector-like. The $U(1)_Y$ symmetry of the Standard Model is an example of a non-vector like gauge symmetry. Now consider the 3-point function $\langle J^\mu(x_1)J^\nu(x_2)J^\rho(x_3)\rangle$ and Fourier transform it to $\Gamma^{\mu\nu\rho}(p_1, p_2, k)$ with $k = -(p_1 + p_2)$, in analogy with the above 3-point function. We require that the result be completely symmetric upon exchanging any of the external, background photons. Imposing this condition, it turns out to be impossible in general to have $p_\mu \Gamma^{\mu\nu\rho} = 0$, much as in the calculations above. The triangle diagram is weighted by $\mathrm{Tr}U(1)_G^3 = \sum_i (q_i^G)^3$. It obviously vanishes for any vector-like symmetry, and it must vanish for any consistent, non-broken, $U(1)_G$ symmetry.

If there are $i = 1 \ldots N$ chiral Fermions, all massless, we can consider $U(1)^N$ symmetries, rotating each one separately. We could also consider $U(N)$ symmetries that rotate them into each other, but we want to consider the possibility that they rotate differently, with different charges q_i. The $U(1)_G$ symmetry above is gauged, so there are $U(1)^{N-1}$ symmetries remaining . Let $U(1)_F$ be some such symmetry, where ψ_i has charge q_i^F. The corresponding current J_F^μ has an ABJ anomaly given by $\partial_\mu J_F^\mu = \mathrm{Tr}U(1)_F U(1)_G^2 \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$, where $\mathrm{Tr}U(1)_F U(1)_G^2 = \sum_i q_i^F (q_i^G)^2$.

• Let's consider the $U(1)_A$ anomaly in $SU(N_c)$ with N_f massless Dirac Fermions. As we discussed earlier, there is an $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$ classical global symmetry, where we can take $U(1)_V$ to act with charge $+1$ on ψ_α and -1 on $\tilde{\psi}_\alpha$, and $U(1)_A$ acts with charge $+1$ on both. If we are discussing $SU(3)_C$ for example, then $U(1)_V = 3U(1)_B$ where $U(1)_B$ is baryon number. There is a $U(1)_V^2 U(1)_A$ anomaly that is the same as in the Abelian case, aside from a multiplicative factor of replacing the $2 \to \mathrm{Tr} U(1)_V^2 U(1)_A = 2N_c N_f$ from the $N_c N_f$ indices of the Fermions that are summed in the loop. Our interest here is instead the anomaly $\partial_\mu J_A^\mu \sim \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr} F_{\mu\nu} F_{\rho\sigma}$ involving the $SU(N_c)$ gauge fields. It comes from the 3-point function involving one $U(1)_A$ current and two $SU(N_c)$ gauge currents. In terms of the triangle diagram, there is a $U(1)_A$ current at one vertex and $SU(N_c)$ gauge fields at the other two. Aside from group theory factors, it is the same triangle diagram as in the Abelian case: we replace $\sum_f q_f^A (q_f^G)^2 \to \sum_f q_f^A \mathrm{Tr}(T^a_{r_f} T^b_{r,f}) \sum_f q_f^A T_2(r_f) \delta^{ab}$. The anomaly is then $\partial_\mu J_A^\mu = k_{AG^2} \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}(F_{\mu\nu} F_{\rho\sigma})$ where $k_{AG^2} = \mathrm{Tr} U(1)_A G^2 = \sum_f q_f^A \mu(r_f)$, with $\mu(r_f) \propto T_2(r_f)$ normalized to be 1 (rather than $\frac{1}{2}$ – the factor of $\frac{1}{2}$ is there in $\mathrm{Tr}\tilde{F}F$) for the fundamental of $SU(N_c)$. In the present example, $k_{ASU(N_c)^2} = 2N_f$. This is the anomaly that explains why there is no 9th light approximate NG boson when $\langle \tilde{\psi}\psi \rangle \neq 0$: it breaks $U(1)_A$, but $U(1)_A$ is already broken by the anomaly (actually, by instantons – this is 't Hooft's solution of the $U(1)$ problem). This is also the anomaly that explains why $\pi^0 \to \gamma + \gamma$ is enhanced.

More generally, we can consider a theory with non-Abelian gauge group G, and some flavor group F that acts on chiral Fermions. For example, we can take $G = SU(N_c)$, with N_f chiral matter fields ψ_α in the $\mathbf{N_c}$, and N_f' in the $\bar{\mathbf{N}}_\mathbf{c}$. The triangle anomaly involving 3 $SU(N_c)$ gauge fields is proportional to $N_f - N_f'$, so the theory is sick unless we consider this case, which is vector-like. Chiral possibilities are e.g. $SU(5)$ with $\psi_\alpha \in \mathbf{10}$ and $\tilde{\psi}_\alpha \in \bar{\mathbf{5}}$; this is what a generation looks like in $SU(5)_{GUT}$. The anomaly involving F and two gauge fields has coefficient $k_{FG^2} = \mathrm{Tr} F G^2$ so it is only non-zero for $F = U(1)_F$, which assigns charge q_i^F to matter field ψ_i, which is in representation r_i of G. The anomaly coefficient is $k_{FG^2} \equiv \mathrm{Tr}(FG^2) = \sum_i q_i^F \mu(r_i)$.

The interpretation is that a G-instanton has $\mu(r_i)$ Fermion zero modes for the Fermion $\psi_{i,\alpha}$, and the non-conservation of J_F^μ is because the instanton carries total charge $\mathrm{Tr}(FG^2)$.

• An anomalous $U(1)_A$, $\psi_{i,\alpha} \to e^{iq_i^A \lambda} \psi_{i,\alpha}$ effectively shifts the G theta angle $\theta_G \to \theta_G - k_{AG^2} \lambda$. Whenever such a broken symmetry is present, the upshot is that θ_G is unphysical – it can be rotated away into the unobservable phases of Fermions! For example, in the standard model the strong CP problem arises because a $SU(3)_C$ θ term leads to an electric dipole moment for the neutron, $d_n \sim e\theta m_q/M_N^2 < 3 \times 10^{-26} ecm$, which leads to $\theta < 10^{-10}$. There is no issue if a quark is massless, since then we get a $U(1)_A$ symmetry and can rotate away any θ. For quarks with mass M, the mass and the anomaly both break $U(1)_A$, so θ becomes physical, but we can still use the anomaly to conclude that the physical quantity is $\bar{\theta} = \theta + \arg\det M$, where M is the quark mass matrix. The strong CP problem is that $\bar{\theta} \leq 10^{-10}$ looks like a fine tuning. If we replace θ with a dynamical axion field $a(x)$, then instantons can generate an effective potential for the analogous $\bar{a}$. This works, with minimum at $\bar{a} \to 0$, if the a is the Nambu-Goldstone boson of a spontaneously broken $U(1)_{PQ}$ Peccei-Quinn symmetry, which has a $\mathrm{Tr} U(1)_{PQ} G^2 \neq 0$ anomaly.

In the Standard Model $SU(3)_C \times SU(2)_W \times U(1)_Y$, we can introduce θ terms for each θ_C, θ_W, θ_Y. The Standard Model Lagrangian preserves classical $U(1)_B$ and $U(1)_L$ global symmetries, where baryon number is $(n_q - n_{\tilde{q}})/3$, and lepton number only acts on the leptons. Both are preserved by $SU(3)_C$, and both are broken by $SU(2)_W$'s anomaly / instantons (and also $U(1)_Y$). The linear combination $U(1)_{B-L}$ is anomaly free. An effect of the $U(1)_{B+L}$ anomaly, and the fact that the chiral quarks do not have mass terms (masses arise from the Higgs Yukawa coupling) is that θ_2 and θ_Y can be rotated, with $\theta_{EM} = (\theta_Y + 18\theta_2)/4$ invariant and physical (see Tong for more on this).