215c, 5/27/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 9 reading: Tong chapter 3 on anomalies.**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• Recall from last time: the wine bottle potential for a charged scalar $\phi = \frac{v+h(x)}{\sqrt{2}} e^{i\pi(x)/f_\pi}$ leads to SSB and if the vev is charged under the gauge group then the gauge field gets a mass $m_A = ev$ from $\mathcal{L} \supset \frac{1}{2}e^2v^2(A_\mu + \frac{1}{ef_\pi}\partial_\mu\pi)^2$ from eating the erstwhile NGB $\pi(x)$. In unitary gauge we set $\pi = 0$. The non-Abelian case is similar, as will be illustrated with an example. As in the global case, the symmetry G is broken to a subgroup H from the vev. The H gauge fields remain massless. The G/H gauge fields get a mass by eating the former NGBs $\pi^a \in G/H$.

• We can generalize the FP procedure for massive gauge fields, taking gauge fixing functional $F^a = \partial_\mu A^{\mu a} - \xi m_A \pi^a$. Then the tree-level propagator for the massive gauge fields is $\frac{i}{p^2-m_A^2}(-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2-\xi m_A^2}(1-\xi))\delta^{ab}$, the NGBs are included with propagator $\frac{i}{p^2-\xi m_A^2}\delta^{ab}$, and the ghosts with propagator $\frac{i}{p^2-\xi m_A^2}\delta^{ab}$. For $\xi \to \infty$, the NGBs and the ghosts are infinitely massive and decouple, and the gauge field propagator sums only over the 3 physical polarizations; this is called unitary gauge because only physical modes propagate. The case $\xi = 1$ is Feynman 't Hooft gauge: the gauge field propagator contains all 4 polarizations, the ghosts subtract off two, and the NGBs add back one.

• 't Hooft proved that renormalizability works (i.e. only a finite number of counterterms are needed) with a massive gauge field if and only if the mass comes from spontaneous symmetry breaking, so the existence of the massive $W_\mu^\pm$ and Z^μ really required the existence of the Higgs field. An anonymous referee for Higgs' paper pointed out that the mechanism implies the existence of a massive scalar particle, now called the Higgs particle H^0, corresponding to the oscillations in the directions transverse to the eaten NGBs. This particle was finally discovered at the LHC in 2011-2013 (our UCSD colleagues contributed to that), with mass $m_{H^0} = 154.18 \pm 0.16 GeV$.

• Let's consider a non-Abelian example of spontaneous symmetry breaking with a gauge symmetry. Consider an $SU(2)$ (or $SO(3)$ – it does not matter for what is said here, though it makes a difference for some non-local observables) gauge theory with matter field $\vec\Phi$ in the adjoint (the $\vec{\cdot}$ denotes the 3 components of the adjoint, which we can think of as a 3d vector). Take $\mathcal{L} = -\frac{1}{4}\vec F_{\mu\nu}\cdot\vec F^{\mu\nu} + \frac{1}{2}\vec{D_\mu}\Phi\cdot\vec{D^\mu}\Phi - V(\vec\Phi\cdot\vec\Phi)$. Take e.g. $V = -\frac{1}{2}m^2\vec\Phi^2 + \frac{\lambda}{4!}(\vec\Phi\cdot\vec\Phi)^2$, so the minimum has $\langle|\vec\Phi|\rangle = v = \sqrt{6m^2/\lambda}$. We can use the $SU(2)$ symmetry to rotate $\vec\Phi$ to point only in $\langle\Phi^a\rangle = \delta^{a,3}v$ and the unbroken symmetry is the $U(1)$ rotation around the $\hat{z}$ axis. If it were not gauged, we would have $G/H = SU(2)/U(1) = S^2$ NGBs. The $U(1)$ associated with A_μ^3 is unbroken and that gauge field is massless. The other two gauge fields eat the π^a NGBs and get mass $m_W = gv$: $\mathcal{L} \supset -\frac{1}{2}g^2v^2\sum_{a=1}^2(A_\mu^a + \frac{1}{gf_\pi}\partial_\mu\pi^a(x))^2$.

As with $L^\pm$ in angular momentum, it is convenient to introduce $A_\mu^\pm$, with charge ± 1 under the unbroken $U(1)$. These massive gauge fields are analogous to the $W^\pm$ of the weak interactions. The difference is that there $SU(2)_W \times U(1)_Y$ is broken to $U(1)_{E\&M}$ by the expectation value of a complex scalar field in the $\mathbf{2}_1$. Here $G/H = SU(2) \times U(1)/U(1) \cong SU(2) \cong S^3$, and the 3 NGBs are eaten by $W_\mu^\pm$ and Z^μ.

For energy $E > m_A$, the UV theory is $SU(2)$ with the adjoint matter field, while for $E < m_A$ the IR theory consists of a photon and we can integrate out the massive $W^\pm$, so the low-energy theory is just free Maxwell theory. The fine structure constant of the low-energy theory is obtained by matching at the scale $\mu = m_A$, so $\alpha_{IR} = \alpha_{UV}(\mu = m_A)$, where $\alpha_{UV}(\mu)$ is the RG running coupling constant.

• This theory contains magnetic monopoles, called 't Hooft Polyakov monopoles. Recall the vortices in the Abelian Higgs theory, if the complex scalar in the vacuum has $\langle|\phi|\rangle = v/\sqrt{2}$, then with compact $U(1)$ there are configurations where e.g. if we take $x + iy = re^{i\theta}$, then we can have $\phi(r \to \infty) = ve^{in\theta}/\sqrt{2}$ with $n \in \mathbf{Z}$. To have finite energy, we need $D_\mu\phi \to 0$ for $r \to \infty$, which requires that A_μ winds around the compact $U(1)$ and gives a configuration with magnetic flux $\int F/2\pi = n$. In the $SU(2) \to U(1)$ theory, there are magnetically charged particles rather than vortices, where the S_∞^2 of space surrounding the particle, for $r \to \infty$ winds around the $G/H = SU(2)$ space of (eaten) NGBs with $\pi_2(G/H) = \mathbf{Z}$ winding number n. Again, $(D_\mu\Phi)^a \to 0$ as $r \to \infty$ for finite energy, and this requires A_μ^a to wind around at infinity, leading to magnetic flux in the unbroken $U(1) \subset SU(2)$. The magnetic charge satisfies the Dirac quantization condition (it would satisfy it even if we allowed for fundamental matter, with $q = \frac{1}{2}$). Such monopoles arise whenever a low-energy $U(1)$ is unified into a non-Abelian group G, and is associated with $\pi_2(G/U(1)) \cong \pi_1(U(1)) = \mathbf{Z}$ (note that $\pi_2(G) = 0$ for any group manifold G, e.g. for $G = SU(2) \cong S^3$, $\pi_2(S^3) = 0$ because $\pi_n(S^m) = 0$ whenever $n < m$). The winding number of the monopole configuration is $n = \frac{1}{8\pi v^2}\int d^2S\hat{n}^i\epsilon^{ijk}\epsilon_{abc}\phi^a\partial_j\phi^b\partial_k\phi^c)$ and the magnetic charge is $q_{mag} = 4\pi n$ with $n \in \mathbf{Z}$. Dirac quantization gives $q_{elec}q_{mag} \in 2\pi\mathbf{Z}$, so q_{mag} is consistent with $q_{elec} \in \frac{1}{2}\mathbf{Z}$. This fits with the fact that matter in $SU(2)$ reps with I half-integral, like the fundamental, have half-integer $U(1)$ charges (like the m quantum number in angular momentum). The configuration with $n = 1$ has $\phi^a = \frac{x^a}{r^2}h(r)$ with $h(r \to 0) \to 0$ (i.e. the SSB turns off in the monopole core and the theory sits at the false vacuum at the origin), and $h(r \to \infty) \to vr$. The energy of the monopole, and hence its rest mass, is bounded by a BPS bound (analogous to the one that we discussed for the action of an instanton) $E \geq \frac{2v}{g^2}|q_{mag}|$. The BPS bound is saturated when $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} = \pm D_i\phi$.

Analogous field configurations exist in 3 Euclidean spacetime dimensions, where they are interpreted as instantons. Polyakov showed that such instantons lead to confinement in 3d.

If the Standard Model $SU(3) \times SU(2) \times U(1)$ is unified into $SU(5)$ or $SO(10)$, there would be such magnetic monopoles, and they would have been produced in the early universe. Inflation can explain why we do not see them.

• Next topic: anomalies of continuous symmetries (discrete symmetries can also have interesting anomalies, will not discuss them). Let's summarize some general results before getting into showing how they arise in specific examples.

1. They arise only in even spacetime dimension d and from massless, chiral matter.
2. In Euclidean spacetime, they are associated with a gauge or background gauge symmetry-violating phase of the functional integral $Z[A^\lambda] = Z[A]\exp(-2\pi i \int_X \alpha(\lambda, A))$, where λ is the gauge transformation parameter and α involves an epsilon tensor (which is why it has the i in Euclidean space, as in our discussion of the theta term in Euclidean space). The anomaly thus cannot arise from real matter representations, which is why it can only arise from massless, chiral matter.
3. They can be understood as the $[d\psi]$ measure in $Z[A] = \int [d\psi]\exp(\frac{i}{\hbar}(S[\psi]+A_\mu J^\mu))$ not respecting the symmetry, or equivalently from $\det(\not{D})$ not respecting the symmetry.
4. Care is needed to see the effect, which is naively zero; it comes from the inability to regulate a divergent quantity in a way that respects all of the symmetries. A non-zero anomaly is an obstruction to writing down a local counter-term that removes the effect. The result is scheme independent.
5. The anomaly arises only at one-loop, from a $\frac{1}{2}(d+2)$-gon diagram, which contributes to a $\frac{1}{2}(d+2)$ point function of currents. The anomaly is topological and is independent of the coupling when the gauge fields are correctly normalized. It is also independent of mass parameters, which again shows that it can only arise from massless fields – for massive fields, the mass can be taken to infinity and the field then decouples.
6. The anomaly is related to index theorems (e.g. Atiyah-Singer) for the Dirac operator in topologically non-trivial field configurations.

7. For 4d, the anomaly involves 3 currents, which can be gauge or global, and the interpretation of the anomaly differs depending on this distinction. The anomaly can be exhibited for a single chiral Fermion, and other cases differ only in the Fermion sum, with trace over the group theory factors, and three symmetrized generators at the vertices. For $U(1)_a U(1)_b U(1)_c$, the anomaly is proportional to $\sum_f (q_f^a q_f^b q_f^c)$, where q_f^a is the charge of Fermion f under $U(1)_a$. For non-Abelian groups G^a replace $q_f^a \to T^a(r_f)$, the generator of the G^a rep r_f of the Fermion, and take the symmetrized trace: the anomaly is proportional to $\mathrm{Tr}(\{T^a(r_f), T^b(r_f)\} T^c(r_f))$. An anomaly involving three gauge fields would make the theory inconsistent (or alternatively Higgsed in the $U(1)$ case).