215c, 5/13/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 7 reading: Tong chapter 2,**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

Also Coleman's Aspect of Symmetry book, chapter on secret symmetry, and Tong chapter 5 for SSB.

• Spontaneous symmetry breaking, aka (Coleman): secret symmetry. The symmetry is not broken at all, but only obscured by the groundstate. E.g. a magnet, where the Hamiltonian is rotationally invariant, but the spins align to point in some particular direction, which we can choose to call $\hat{z}$. This seems to break the symmetry, e.g. the $SO(3)$ rotations down to the little group $U(1)$ of rotations around the z-axis. There are two broken generators of $SO(3)$ and the broken elements of the group live in $SO(3)/SO(2) \cong S^2$, corresponding to rotating $\hat{z}$ to point anywhere on the unit sphere. Spontaneous symmetry breaking is the statement that there is a symmetry of interactions, respected by $\mathcal{L}$, which is disrespected by the groundstate, so $Q|0\rangle \neq 0$ where Q is the charge for a broken generator (e.g. the broken rotation generators). The symmetry is actually only obscured – the technical name is "non-linearly realized" – rather than broken, in that it still has non-trivial consequences. Among those consequences is, if a symmetry G is broken to a subgroup H, then there are massless bosons, called Nambu-Goldstone bosons (NGBs) that live on the space G/H, e.g. the S^2 in the example above. Not only are the bosons massless – they cannot have any potential at all: their interactions can only be via derivatives of the fields.

As a concrete class of examples, consider a theory of N real scalar fields $\Phi^{i=1\ldots N}$, with $\mathcal{L} = \frac{1}{2}\sum_i \partial_\mu \Phi^i \partial^\mu \Phi^i - \frac{A}{2}\sum_i \Phi^i\Phi^i - \frac{B}{4}(\sum_i \Phi^i\Phi^i)^2$. We take $B > 0$, so the energy will be bounded below. If $A > 0$, then the vacuum is at the origin, $\Phi^i = 0$. If $A < 0$ then the origin is a local maximum, and the true minimum is at $\sum_i \Phi^i\Phi^i = -A/B \equiv v^2$, which is an S^{N-1}. The mass matrix is $M^2_{ij} = \partial^2 V/\partial\Phi^i\partial\Phi^j|_{min} = 2B\Phi^i\Phi^j$ which has one non-zero eigenvalue and $N-1$ zero eigenvalues. The global symmetry is $G = SO(N)$ and we can rotate the vacuum expectation value to e.g. $\Phi^i_0 = v\delta^{iN}$, and then the little group of unbroken rotations is $H = SO(N-1)$, and $G/H = S^{N-1}$ has radius v. The G/H fields are massless because we can have arbitrarily long-wavelength fluctuations $\delta\Phi^i$ around Φ^i_0. The low-energy theory is called a non-linear sigma model, with target space S^{N-1}, and the leading order Lagrangian is $\mathcal{L} = \sum_i \partial_\mu\Phi^i\partial^\mu\Phi^i$ with the constraint $\sum_i \Phi^i\Phi^i = v^2$; this can also be written as $\mathcal{L} = \frac{1}{2}g_{ab}(\phi)\partial_\mu\phi^a\partial^\mu\phi^a$, where $\phi^a \in S^{N-1}$ with metric $g_{ab}(\phi)$. For

$N = 4$, this can be equivalently written as a theory with $G = SU(2)_L \times SU(2)_R$, with $\tilde{\Phi} \equiv \sum_i \Phi^i \sigma^i \in (2,2)$; then $\langle \tilde{\Phi} \rangle = v\mathbf{1}$ breaks $G \to H = SU(2)_D$, and $G/H \cong SU(2) \cong S^3$.

• Consider the general case of a G global symmetry, which acts on the fields as $\delta\psi_n = e^{iT_n^a \alpha^a}\psi_n$, where α^a are constants. This leads to conserved Noether currents $J^{\mu,a}$, with $\partial_\mu J^{\mu,a} = 0$, with conserved charge $Q^a = \int d^3x J^{0,a}$. The charges generates the G transformation and all operators form representations, $i[Q^a, \mathcal{O}_n] = (T^a)_n^m \mathcal{O}_m$. In the unbroken case, a correlation function $\langle \prod_i \mathcal{O}_i(x_i) \rangle$ is only non-zero if the product of operators gives the trivial, neutral, identity representation. The symmetry is instead spontaneously broken if some Q^a (those of some subgroup H) do not annihilate the vacuum, $Q^a|0\rangle \neq 0$. This happens iff there is some scalar operator $\mathcal{O}$ (it can be a composite, e.g. $\bar{\psi}\psi$), with non-zero Q^a charge, $[Q^a, \mathcal{O}] \neq 0$, has a non-zero vacuum expectation value (vev), $\langle \mathcal{O} \rangle \neq 0$. Since $\mathcal{O}$ is charged, $\mathcal{O} \sim [Q^a, \mathcal{O}]$ so its non-zero vev is only possible if Q^a does not annihilate the vacuum. Because Q^a is a symmetry of the theory, and in particular the Hamiltonian, $[H, Q^a] = 0$, so $Q^a|0\rangle$ and $|0\rangle$ are degenerate in energy. Q^a is actually non-normalizable because of all this, but this does not affect its commutators. Let's focus on the currents.

The statement that $Q^a|0\rangle \neq 0$ means that $J^{a,\mu}|0\rangle$ cannot be zero. Instead, it creates a state: $\langle \phi^a(p)|J^{b,\mu}(x)|0\rangle = i\delta^{ab}p^\mu f e^{ip\cdot x}$. Current conservation $\partial_\mu J^\mu$ requires the state to be massless, $p^2 = 0$. Recall from Fall that we normalized our creation operators, and thus our states, such that, in D spacetime dimensions, $\langle k'|k\rangle = (2\pi)^D (2\omega_k)\delta^{D-1}(\vec{k} - \vec{k}')$, i.e. the state $\langle \phi(k)|$ has mass dimension $\frac{1}{2}(2 - D)$. The current $J^{b,\mu}$ has mass dimension $D - 1$, so the charge Q is dimensionless. Thus $\Delta(f) = \frac{1}{2}(D - 2)$, which has the units of a scalar field. In fact, $f \sim v$ is the vacuum expectation value of some scalar, which sets the size radius of the G/H target space.

Aside: the fact that $\Delta(f) = 0$ in $D = 2$ is related to the fact that scalars have *log* rather than power-law correlation functions. There is a theorem (Coleman) that continuous symmetries cannot be spontaneously broken in $D \leq 2$ non-compact space or time dimensions. This is analogous to the theorem mentioned earlier that symmetry breaking, even for a discrete symmetry, is impossible in $D = 1$ QM. For $D = 2$, discrete symmetry breaking is possible, but not continuous.

• Another way to see it: consider $\langle J^{a,\mu}(x)\mathcal{O}(y)\rangle$. The Ward identity (recall a HW from Fall) says that the current is conserved in the correlator up to a contact term: $\partial_x^\mu \langle J^{a,\mu}(x)\mathcal{O}(y)\rangle = \delta(x_0 - y_0)\langle [J^{0,a}(x), \mathcal{O}(y)]\rangle = i\delta^D(x - y)\langle \delta^a \mathcal{O}(y)\rangle$, which is non-zero by assumption. If we Fourier transform, $\langle J^{a,\mu}(x)\mathcal{O}(y)\rangle = \int d^4k \sigma^a(k^2)\theta(k_0)k_\mu e^{ik(x-y)}$, see that $\sigma^a(k^2)$ has to have a pole at $k^2 = 0$: $\sigma^a \sim \langle \delta^a \mathcal{O} \rangle / k^2$. Such a pole is associated with a massless particle $\langle \phi^a(k)|$.

A simple and illustrative toy model, in the context of a $U(1)$ global symmetry, is a free massless, compact scalar ϕ with $\phi \sim \phi + 2\pi f$. There is a shift symmetry $\delta\phi = \alpha$, so $\delta\phi \equiv \frac{\delta}{\delta\alpha}\phi = 1$; because this has non-zero expectation value $\langle \delta\phi \rangle = 1$, the shift symmetry is spontaneously broken. The conserved current is $J^\mu = f\partial^\mu\phi$, where f has dimensions of energy, and indeed $i[j^0(\vec{x}, t), \phi(\vec{y}, t)] = \delta^3(x - y)\delta\phi$. The current J^μ creates the massless state $|k\rangle = a^\dagger(k)|0\rangle$, with $\langle k|J^\mu(x)|0\rangle \sim fk^\mu e^{ikx}$. Integrating $\int d^3x J^0$ in this expectation value is peaked around $\vec{k} = 0$ and the result is that the charge is non-normalizable, $||Q|0\rangle||^2 \to \infty$.

An example that reduces to the above in the IR is $\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^* \partial^\mu \Phi - V(|\Phi|)$ for the case where V has a minimum for $\langle \Phi^* \Phi \rangle \equiv v^2 \neq 0$. Take $\Phi = \rho e^{i\phi/f}$, then $\mathcal{L} \to \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{\rho^2}{2f^2}\partial_\mu\phi\partial^\mu\phi - V(\rho)$, and we assume that the potential has a minimum at $\rho = v$. The fluctuation $\delta\rho$ is massive, while ϕ is massless. The global symmetry that gives Φ a phase shifts ϕ. This is called a non-linear realization of the symmetry (even though the shift of ϕ is linear), because $\phi \sim \ln \Phi$. The periodicity in the phase, associated with $U(1)$ being compact, yields periodicity in ϕ.

• Let's consider the symmetries of $SU(N_c)$ with N_f massless Dirac Fermions in the fundamental of $SU(N)$. There are classical global symmetries $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$, and we will write the Fermions as $\psi_\alpha \in (N_c; N_f, 1)_{1,1}$ and $\tilde{\psi}_\alpha \in (\bar{N}_c; 1, \bar{N}_f)_{-1,1}$, where $\alpha = 1, 2$ is the chiral Fermion spinor index; $\psi^\dagger$ and $\tilde{\psi}^\dagger$ are right-handed, with spinor index $\dot{\alpha} = 1, 2$, and in the conjugate representations. The subscripts denote the $U(1)$ charges. The $U(1)_V$ symmetry acts on the Dirac Fermions via $j^\mu_V = \bar{\Psi}\gamma^\mu\Psi$, and $U(1)_A$ acts via $j^\mu_A = \bar{\Psi}\gamma^\mu\gamma^5\Psi$, and $SU(N_f)_{L,R}$ act as $j^{\mu,a}_{SU(N_f)_{L,R}} = \bar{\Psi}\gamma^\mu T^a P_{L,R}\Psi$, where $P_{L,R} = \frac{1}{2}(1 \mp \gamma^5)$. As we will discuss soon, $U(1)_A$ is violated by instantons, and this fact shows up in a triangle diagram with $U(1)_A$ current insertion at one vertex, and $SU(N_c)$ gauge current operators inserted at the other two.

Consider the gauge and Lorentz invariant quantity $\psi^{f,c}_\alpha \tilde{\psi}^{\tilde{g}}_{c,\beta}\epsilon^{\alpha\beta}$, where the gauge indices are contracted to give a gauge singlet, and the Lorentz spinor indices are contracted to give a scalar, but the flavor indices are not contracted; suppressing the indices, this is $\psi\tilde{\psi}$ in the $(N_f, \bar{N}_f)_{0,2}$. For N_f just below the asymptotic freedom bound, the IR phase is expected to be an interacting conformal field theory with $\beta(\alpha_*) = 0$. For lower values of N_f, the original theory becomes too strongly coupled in the IR to admit the CFT phase, and instead the operator $\psi\tilde{\psi}$ gets a non-zero vev, which spontaneously breaks the global symmetry G to the little group H, i.e. the subgroup that is preserved by the expectation value. Goldstone's theorem says that spontaneously breaking $G \to H$ leads to massless, Nambu-Goldstone bosons, aka pions, in the coset G/H. In the present case, classically, $G = SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$ and $H = SU(N_f)_{L+R\equiv D} \times U(1)_V$, so the classical NGB space is $G/H = SU(N_f) \times U(1)_A$. This led to a famous puzzle. The resolution of the puzzle is that $U(1)_A$ is actually not a symmetry – it has an anomaly and is explicitly violated by instantons. So the quantum G/H space is actually just $SU(N_f)$.

ended here

• This was first observed in real-world physics, which guided theorists to the understanding mentioned above. Approximate global symmetries, where the symmetry is explicitly broken by some small amount, are also useful. The small explicit symmetry breaking can lift the G/H degeneracy, e.g. tilting the sombrero, giving a vacuum where the NGBs have a small mass that is related to the amount of symmetry breaking. They are then called pseudo NGBs and this played a big role in the development of particle physics and QFT.

The $N_f = 3$ light quark flavors (u, d, s) have an approximate $SU(3)_L \times SU(3)_R$ global symmetry, which is respected by the $su(3)_c$ strong force but broken explicitly by the nonzero masses, and by $su(2)_W \times u(1)_Y$ and by $u(1)_{E\&M}$. This approximate global symmetry is spontaneously broken by $\langle \psi\tilde{\psi}\rangle \neq 0$, leading to approximate NGBs in the adjoint of the diagonal $SU(3)_F$. There are indeed 8 light pions which fit this perfectly:

$$\begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{-\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -2\frac{\eta^0}{\sqrt{6}} \end{pmatrix} \in \mathbf{8}.$$

Aside: the baryons, including the proton and the neutron and others, also form $SU(3)_F$ representations and this was the original 8-fold way of Gell-Mann, which he used to predict the existence, and the mass, of a baryon that is now understood to be made up from three strange quarks in the **10** of $SU(3)_F$ and with spin $j = 3/2$. Note that this is completely symmetric in the $SU(3)_F$ labels and in the spin, and this fits with Fermi statistics because it is completely antisymmetric in $su(3)_c$ to get something color neutral.

If $U(1)_A$ were a symmetry, there would have to be a 9th pseudoscalar (since it is P odd) meson; the candidate observed particle is called the η', but it is too massive to be considered an approximate NGB. The resolution is that $U(1)_A$ is not a symmetry, as already mentioned, and this gives the η' a large mass compared to the light pions. The pions are not massless because the global symmetries are explicitly broken by the non-zero quark mass terms; approximate values are $m_u \approx m_d \approx 0.307 GeV$, $m_s \approx 0.490 GeV$, and approximate formulae for the meson masses from this explicit breaking would give $m_{\eta',wrong} \approx 355 MeV$ whereas $m_{\eta',actual} \approx 958 MeV$.

• The G/H space $SU(N_f)_D$ has non-trivial topology: it contains a S^3 so $\pi_3(G/H) = \mathbf{Z}$ for $N_f \geq 2$. For $N_f \geq 3$ it also contains a S^5, so $\pi_5(G/H) = \mathbf{Z}$ for $N_f \geq 3$. The S^3 means that there can be solitonic particle configurations, where space and the point at infinity, are wound around the S^3 target; these are called Skyrmions, and it turns out that they have the right quantum numbers to give the baryons of the original UV theory of quarks and gluons, now realized as solitons on the space of pions. The S^5 plays a role in giving what is known as the Wess-Zumino-Witten interaction of the low-energy theory. If there is time, this will be discussed in the context of 't Hooft anomaly matching.