215c, 5/6/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 6 reading: Tong chapter 2.**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• Last time: sketched how to get the beta function from $g_B = g\mu^{\epsilon/2} Z_{A^3}/Z_{A^2}^{3/2}$ where $\mathcal{L} \supset -\frac{1}{2}\mathrm{Tr}(Z_{A^2}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - igZ_{A^3}(\partial_\mu A_\nu - \partial_\nu A_\mu)[A_\mu, A_\nu] + \ldots)$, and $A_B^\mu = \sqrt{Z_{A^2}}A^\mu$. Manifest gauge invariance would require $\mathcal{L} \supset -\frac{1}{2}Z\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$ with $F_{\mu\nu} = i[D_\mu, D_\nu]$ and $D_\mu = \partial_\mu - igA_\mu$, i.e. $Z_{A^2} = Z_{A^3} \equiv Z$, but the gauge fixing of course breaks manifest gauge invariance (which gives an opportunity for a cross-check, to see that it is recovered at the end of the calculation when the dust settles). In particular, we gauge fixed with gauge fixing condition $F^a = \partial_\mu A^\mu$, which led to $\mathcal{L}_{ghost} = \partial_\mu \bar{c}^a D_\mu c^a$ which is obviously not gauge invariant because of the ordinary derivative. The ghosts have the cubic vertex with a gauge field, weighted by f^{abc}, as expected for a field in the adjoint; but they do not have the usual seagull quartic vertex of a scalar. There are three one-loop diagrams in pure YM that contribute to the gauge field propagator, giving $Z_{A^2} - 1 = \delta_{A^2}$. If we include charged matter, there are additional diagrams with the scalars or Fermions running in the loop. There are three IPI one-loop diagrams contributing to the 3-gauge field coupling and hence $Z_{A^3} - 1 = \delta_{A^3}$ (one loop of gauge field with 3 cubic vertices, one with one cubic and one quartic vertex, and one ghost loop). Again, there are additional diagrams from charged Fermions or scalars in the loop. Summing all these contributions leads to $\beta(g) = -\frac{g^3}{48\pi^2}(11C(adj) - 4T(r_{DF}) - T(r_{CS})) + \mathcal{O}(g^5)$ where the last expression includes the effect of Fermions in representation r_{DF} and scalars in representation r_{CS}. Here DF stands for Dirac Fermion and we should divide the contribution to the beta function in half if the Fermions are chiral; likewise CS stands for complex scalars and we should divide the contribution in half if the scalars are real. Again, the remarkable point is that the coefficient can be negative, which cannot happen in 4d for any theory other than non-Abelian gauge theories. If there is not too much matter, $\beta < 0$, leading to asymptotic freedom in the UV and strong coupling in the IR.

• There is a different gauge choice that requires fewer diagrams, and is conceptually interesting. It is called background field gauge. It is like having your cake and eating it too. On the one hand, we need to break gauge invariance, fixing a gauge, to write the gauge field propagator. Then we have to wait until the dust settles in the calculation to see that the final result is gauge invariant, e.g. the cancellation of the ξ parameter mentioned above. Background field gauge is a way to break gauge invariance, but still have manifest background gauge invariance in the calculation. Global currents can be coupled to background gauge fields sources. For a gauge symmetry, there are instead dynamical gauge fields that we integrate over in the path integral. But we can still couple a global component of the gauge symmetry to a background source. We mentioned an example of this earlier, with Schwinger's calculation of charged particle production in an external electric field. As was the case there, we include vertices where the dynamical fields couple to the external source. There are then two types of gauge transformation: that of the dynamical gauge field, and that of the background source. We can choose gauge fixing such that they violate gauge invariance but preserve the sum of gauge plus background gauge invariance.

We can phrase it as above with the gauge fixing condition $F(A)$, where we had $\mathcal{L}_{g.f.} = -\frac{1}{2\xi}F^aF^a$ and $\mathcal{L}_{ghost} = \bar{c}^a\frac{\partial F^a}{\partial A^b_\mu}(D_\mu c)^b$. Introduce a classical background $\bar{A}^a_\mu$ and corresponding $\bar{D}_\mu = \partial_\mu - ig\bar{A}_\mu$. Then we take $F^a = (\bar{D}^\mu(A-\bar{A})_\mu)^a$ and $\mathcal{L}_{ghost} = -\bar{D}^\mu\bar{c}D_\mu c$. We now take the gauge transformations δ_G to act on A_μ and the fields as usual, but take $\bar{A}_\mu$ to be invariant. There are also background gauge transformations δ_{BG}, which only act on $\bar{A}_\mu$ and leave A_μ and all the fields invariant. The gauge fixing terms of course break δ_G (that's their job) and they also break δ_{BG}. But everything preserves the combination δ_{G+BG}. For example, $\delta_G A_\mu = D_\mu\omega_G$ and $\delta_G\bar{A}_\mu = 0$ and $\delta_{BG}A_\mu = 0$ and $\delta_{BG}\bar{A}_\mu = \bar{D}_\mu\omega_{BG}$. Then taking $\omega_G = \omega_{BG}$ get $\delta_{G+BG}(A-\bar{A})_\mu = (D_\mu - \bar{D}_\mu)\omega = -ig(A-\bar{A})_\mu\omega = ig\omega(A-\bar{A})_\mu$ (the last step can be seen by writing out $\omega = \omega^aT^a$ and $A - \bar{A} = (A-\bar{A})^bT^b$ and $(T^b_{adj})^{ac} = -if^{bac} = if^{abc} = -(T^a_{adj})^{bc}$). So $(A-\bar{A})_\mu$ transforms in the adjoint, so F^a transforms in the adjoint. Everything ends up being invariant under δ_{G+BG}.

Upon doing the functional integral over the dynamical gauge fields, the effective action, as a function of the background fields, must then be invariant under background gauge symmetry (otherwise we would say that the symmetry is anomalous - i.e. violated by quantum effects – and we will see that this happens for some global symmetry, but gauge symmetries cannot be anomalous). The background gauge symmetry then ensures that $Z_{A^2} = Z_{A^3} = Z_{A^4} \equiv Z$; this is the huge simplification. So $g_B^2 = Z^{-1}g^2\mu^\epsilon$ and it's now possible to compute the beta function simply from the gauge field 2-point function diagrams that contribute to Z_{A^2}. Let's write $A_\mu - \bar{A}_\mu \equiv \delta A_\mu$ (where δA_μ can be thought of as the quantum fluctuations). $\bar{A}_\mu$ can be roughly thought of as an IR part, and δA_μ are the UV fluctuations. We will do the functional integral over the δA_μ, and get an effective action for the $\bar{A}_\mu$. In terms of the diagrams, the internal gauge field propagators are δA_μ, and the external ones are $\bar{A}_\mu$. We take $\delta_G\bar{A}_\mu = 0$ and $\delta_G(\delta A_\mu) = \delta_G A_\mu =$

$D_\mu\omega = (\partial_\mu - igA_\mu)\omega = \bar{D}_\mu\omega - ig\omega A_\mu$. The gauge fixing condition is $F = \bar{D}^\mu\delta A_\mu$, so $S_{g.f.} = \frac{1}{2\xi}\int d^4x(\bar{D}^\mu\delta A_\mu)^2$ and the FP determinant gives $\int dc d\bar{c}\exp(-S_{ghost})$ with $S_{ghost} = \int d^4x \mathrm{Tr}[\bar{D}_\mu\bar{c}\bar{D}^\mu c + igc^\dagger[\bar{D}^\mu\delta A_\mu, c]$. Note that the ghost has the same propagator as before, and it has a cubic vertex that is similar to the one before, but now it is coupling to the external background field $\bar{A}^a_\mu$. There is a similar cubic vertex where it couples to the dynamical gauge field. Finally, there two seagull-like vertices, coupling incoming and outgoing ghosts to two gauge fields: the first term contributes such a term involving two background gauge fields, and the second term contributes involving one background and one dynamical gauge field.

We can obtain $Z = Z_{A^2}$, and thus the beta function, from the diagrams where we couple to two external gauge fields $\bar{A}^a_\mu$ and $\bar{A}^b_\nu$. At one loop, in pure YM, we have 4 diagrams: the 3 from before, plus one with a ghost loop and one seagull vertex to the background gauge fields. There are some nice cancellations between the gauge fixing term and the terms in $\mathcal{L}_{YM}$ for $\xi = 1$, but the result is of course ξ independent in any case. An equivalent way to think about the calculation, discussed in detail in Tong's lecture notes is seen from the functional integral as follows (going back to the normalization of the gauge fields where g only appears in front of the gauge kinetic terms)

$$e^{-S_{eff}[\bar{A}]} = \int [D\delta A][Dc][D\bar{c}]e^{-S[\bar{A},\delta A,c,\bar{c}]} = \frac{\det \Delta_{ghost}}{\sqrt{\det \Delta_{gauge}}}e^{-\frac{1}{2g^2}\int d^4x \mathrm{Tr}\bar{F}^{\mu\nu}\bar{F}_{\mu\nu}}$$

$\Delta^{\mu\nu}_{gauge} = -\bar{D}^2\delta^{\mu\nu} + 2i[\bar{F}^{\mu\nu}\cdot]$, $\Delta_{ghost} = -\bar{D}^2$. So

$$S_{eff}[\bar{A}] = \frac{1}{2g^2}\int d^4x \mathrm{Tr}\bar{F}^{\mu\nu}\bar{F}_{\mu\nu} + \tfrac{1}{2}\mathrm{Tr}\ln\Delta_{gauge} - \mathrm{Tr}\ln\Delta_{ghost}.$$

See the Tong notes for a nice, detailed calculation of these quantities, and how they lead to $S_{eff} \supset \frac{1}{2g^2(\mu)}\int d^4x \mathrm{Tr}\bar{F}^{\mu\nu}\bar{F}_{\mu\nu}$ with $g^{-2}(\mu) = g^{-2} - \frac{11}{3}\frac{C(adj)}{16\pi^2}\ln(\frac{\Lambda^2}{\mu^2})$.

• Let's go back to the intuitive picture of the beta function being associated with the vacuum screening or anti-screening charges. There is a way to reproduce the one-loop beta function from a direct calculation along these lines. I will briefly sketch it, and you can find more details in Preskill's online lecture notes (linked in the class' webpage). Write the one-loop beta function as giving $g^2(\mu) = g_0^2/(1 - g_0^2\frac{b_1}{16\pi^2}\ln(\Lambda^2/\mu^2))$, which can be interpreted as $\epsilon^{-1}_{dielectric} = \mu_{permeability} = (1 - g_0^2\frac{b_1}{16\pi^2}\ln(\Lambda^2/\mu^2))$, so the susceptibility is $\chi = (\mu_{permeability} - 1)/\mu_{permeability} \approx -\frac{b_1}{16\pi^2}g_0^2\ln(\Lambda^2/\mu^2)$. The idea is to turn on an external magnetic field and read off χ from $dU = -MdB = -\chi BdB$ i.e. $U = -\frac{1}{2}\chi B^2$. For a non-Abelian theory, we can restrict to a $U(1)$ subgroup and then reassemble at the end. Turning on the external B field leads to Landau levels for the charged matter fields, e.g. a spin 0, charged massless particle has $E^2 = (\vec{p} - e\vec{A})^2 = p_z^2 + (2n+1)eB$ upon quantizing the rotation in the (x, y) plane. Adding these contributions and regulating the dp_z integral leads to $\chi_{diamagnetic} = -\frac{1}{48\pi^2}e^2 \ln \Lambda^2$, where Λ is a UV cutofff in the momentum integral. For Fermions it is the $E < 0$ rather than the $E > 0$ levels that are occupied, so they contribute to $\chi_{diagmagnetic}$ with opposite sign, i.e. there is a factor of $(-1)^F$, giving the usual loop-weighting factor. To see the possibility of paramagnetic contributions, we need to include the energy contribution from the particle's magnetic moment, i.e. $E^2 = (\vec{p} - e\vec{A})^2 - g_{ratio} e\vec{B} \cdot S_z$, giving (setting $g_{ratio} = 2$ and replacing $e \to Qe$ for a particle of charge Q). The result for a complex field is $\chi = \frac{1}{16\pi^2}(-1)^F(-\frac{1}{3} + 4S_z^2)e^2 \ln \Lambda^2$. So $b_1 = Q^2(-1)^F(4S_z^2 - \frac{1}{3})$. We now sum over all the charged matter fields Q_i, and we can reassemble back into the non-Abelian group factors by replacing $\sum_i Q_i^2 \to \mathrm{Tr}_r(T^{u(1)}T^{u(1)}) = T_2(r)$ (e.g. for $SU(N_c)$ we can pick T to be any generator, e.g. T_3 for any $SU(2)$ subgroup, to get $T_2(fund) = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}$ and $T_2(adj) = C(adj) = (1)^2 + (1)^2 + 2(N_c - 2)T_2(fund) = N_c$) and use $S_z = 0$ for scalars, $S_z = \pm\frac{1}{2}$ for Fermions, and $S_z = \pm 1$ for spin 1 (i.e. the gauge fields – for the case of the gauge fields, there is an extra factor of $\frac{1}{2}$ because they are real). This reproduces $b_1 = \frac{1}{3}(11C(adj) - 4T(r_{DF}) - T(r_{CS}))$. Asymptotic freedom is a consequence of the large magnetic moments of the spin 1 charged matter from the non-Abelian gauge fields.

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• Write the one-loop beta function as $\frac{d}{d \ln \mu}(-8\pi^2 g^{-2}) = 16\pi^3 g^{-3}\beta(g) \equiv -b_1$, which integrates to $e^{-8\pi^2/g^2(\mu)} = (\frac{\Lambda}{\mu})^{b_1}$, with $b_1 = \frac{1}{3}(11C(adj) - 4T(r_{DF}) - T(r_{CS}))$. For example, for $SU(N_c)$ gauge field with N_f Dirac flavors in the fundamental this gives $b_1 = \frac{1}{3}(11N_c - 2N_f)$. The theory is asymptotically free if $N_f < \frac{11}{2}N_c$. Aside: with supersymmetry we add new fields: gauginos which are adjoint chiral Fermions (2-component, to match the two polarizations of the gauge fields) and complex scalars in the $N_c + \bar{N}_c$ to match the d.o.f. of the Fermion flavors, so $b_1 = \frac{1}{3}(11N_c - 2N_f - 2N_c - N_f) = 3N_c - N_f$, so the theory is asymptotically free if $N_f < 3N_c$. If we go to two loops, then $\frac{d}{d \ln \mu}(-8\pi^2 g^{-2}) \equiv -b_1 + b_2 g^2$, and it turns out that the 2-loop beta function coefficient b_2 is positive. There is then a possibility of having $\beta(g_*) = 0$, at $g_*^2 = b_1/b_2$. Draw the picture. If the theory is just barely asymptotically free, then b_1 can be small and b_2 large, so that this fixed point can be weakly coupled, and then we can trust the perturbative calculations. Lattice gauge theory gives a way to determine the conformal window outside of perturbation theory, and this has been part of Prof. Julius Kuti's research program.

• If we plot the one-loop $2\pi/\alpha = 8\pi^2/g^2$ as a function of $\ln\mu$, the one-loop running gives a straight line, with slope b_1; asymptotically free couplings have $b_1 > 0$ and non-asymptotically free ones have $b_1 < 0$. Let's consider the effect of a field of mass m. For energies $\mu > m$, the massive field contributes in the loop and there is some beta function $\beta_H(g)$. For energies $\mu < m$, the massive field decouples (this is not obvious in mass independent renormalization schemes, e.g. $\overline{MS}$), and the low-energy theory has beta function $\beta_L(g)$. The coupling g is of course continuous across the scale m, and this gives a threshold matching relation for the dynamical scale. To one-loop, this gives $(\frac{\Lambda_H}{m})^{b_{1,H}} = (\frac{\Lambda_L}{m})^{b_{1,L}}$, i.e. $\Lambda_L^{b_{1,L}} = m^{b_{1,L}-b_{1,H}}\Lambda_H^{b_{1,H}}$, e.g. for $SU(N_c)$ with N_f matter fields in the fundamental, if we give a mass to one flavor, then the slope changes from $b_{1,H} = \frac{1}{3}(11N_c - 2N_f)$ to $b_{1,L} = \frac{1}{3}(11N_c - 2(N_f-1))$ and $\Lambda_L^{b_{1,L}} = m^{2/3}\Lambda_H^{b_{1,H}}$. Note that we need $\mu, m > \Lambda$ and thus $\Lambda_L > \Lambda_H$; this reflects the fact that, by decoupling a matter field, the low-energy theory is more asymptotically free, and thus more strongly coupled in the IR.

• Let's consider the SM: the gauge group is $SU(3)_C \times SU(2)_W \times U(1)_Y$ with three generations of chiral fermions in the $(3,2)_{1/3} + (\bar{3},1)_{-4/3} + (1,2)_{-1} + (1,1)_2$. The Higgs field is a complex scalar in the $(1,2)_1$. For $U(1)_Y$ the slope is $b_1 = -\frac{4}{3}\mathrm{Tr}_{DF}Q^2 - \frac{1}{3}\mathrm{Tr}_{CS}Q^2$ so this gives $-\frac{4}{3}\cdot 3\cdot(3(1/3)^2 + (3/2)(-4/3)^2 + \frac{3}{2}(2/3)^2 + \frac{1}{2}(2)(-1)^2 + \frac{1}{2}(2)^2 - \frac{1}{3}(2)(1)^2$. The slopes for $SU(3)\times SU(2)\times U(1)_Y$ are then found to be $(7, \frac{19}{6}, -\frac{41}{3})$. The three couplings approximately, but do not quite meet at $\mu \sim 10^{15}GeV$. This is some approximate evidence for grand unification, and the fact that they miss can be modified by adding additional matter fields at higher energy. E.g. in the MSSM get slopes $(3, -1, -33/5)$. and better convergence of the couplings at $M_{GUT?} \sim 10^{15}GeV$.