215c Homework exercises 6, Spring 2020, due May 29

1. Practice computing anomaly coefficients:

As will be discussed next week, anomalies in four spacetime dimensions are proportional to $\mathrm{Tr}(\{G_1, G_2\}G_3)$, where $G_{1,2,3}$ are any symmetries and Tr runs over all of the massless, left-handed, chiral Fermions. If a Fermion is in representation (r_1, r_2, r_3) of groups G_1, G_2, G_3, then $\mathrm{Tr}(G_1 G_2 G_3) \to \mathrm{Tr}(\{T^{a_1}_{r_1}, T^{a_2}_{r_2}\}T^{a_3}_{r_3})$ and the anti-commutator ensures that the result is symmetric in (a_1, a_2, a_3). For G_i that differ, the trace is over the tensor product space, and thus factorizes into a product of the separate traces. For the case of all Abelian groups, $\mathrm{Tr}U(1)_1 U(1)_2 U(1)_3 = \sum_i q_{1,i} q_{2,i} q_{3,i}$ where $q_{a,i}$ is the $U(1)_a$ charge of field i. For the case of two non-Abelian generators and one $U(1)$ generator, $\mathrm{Tr}G^a G^b U(1) = \delta^{ab} \sum T_2(r_i) q_i$, where $T_2(r)$ is the quadratic index, $\mathrm{Tr}(T^a(r)T^b(r)) = T_2(r)\delta^{ab}$. For the case of three non-Abelian generators, $\mathrm{Tr}G^3$ is associated with the cubic Casimir of the group, and is proportional to the $d^{a,b,c}$ symbol, $\mathrm{Tr}(\{T^a_r, T^b_r\}T^c_r) \equiv A(r) d^{abc}$, where d^{abc} are some completely symmetric constants that depend on the group, but not the representation r, and we can define $A(r) = 1$ for a fundamental of $SU(N)$. Fact: only $SU(N)$ groups, with $N \geq 3$ have a non-zero cubic Casimir, i.e. a non-zero d^{abc}. Some methods to compute $d(r)$ are $d(r_1 + r_2) = d(r_1) + d(r_2)$ and $d(r_1 \times r_2) = d(r_1)\dim(r_2) + d(r_2)\dim(r_1)$, and also it is enough to consider how the rep decomposes under a $SU(3) \supset SU(N)$ where $\mathbf{N} \to \mathbf{3} + (N-3)(\mathbf{1})$.

(a) Recall the $SU(N_c) \times [SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A]$ discussed in lecture for $SU(N_c)$ gauge theory with N_f massless Dirac Fermion flavors. Compute each of the following, remarking in particular about which are zero vs non-zero: $\mathrm{Tr}SU(N_c)^3$, $\mathrm{Tr}SU(N_c)^2 SU(N_f)_L$, $\mathrm{Tr}SU(N_c)^2 U(1)_V$, $\mathrm{Tr}SU(N_c)^2 U(1)_A$, $\mathrm{Tr}SU(N_c)SU(N_f)^2_L$, $\mathrm{Tr}SU(N_c)U(1)^2_Y$, $\mathrm{Tr}SU(N_f)^2_L U(1)_V$, $\mathrm{Tr}SU(N_f)^2_L U(1)_A$, $\mathrm{Tr}U(1)^3_V$, $\mathrm{Tr}U(1)^3_A$, $\mathrm{Tr}U(1)^2_V U(1)_A$.

(b) Refer to the lecture on May 11 to see the $SU(3)_C \times SU(2)_W \times U(1)_Y$ representations of a generation of chiral, left-handed matter fields in the Standard Model. Verify that all of its anomalies involving the gauge fields vanish: $\mathrm{Tr}SU(3)^3_C = 0$, $\mathrm{Tr}SU(3)^2_C U(1)_Y = 0$, $\mathrm{Tr}SU(2)^2_W U(1)_Y = 0$, $\mathrm{Tr}U(1)^3_Y = 0$.

(c) Consider an $SU(N)$ gauge theory with one chiral Fermion in the $\mathbf{\frac{1}{2}N(N+1)}$, one in the $\overline{\mathbf{\frac{1}{2}N(N-1)}}$, N_A in the adjoint, $N_{f,L}$ in the $\mathbf{N}$ and $N_{f,R}$ in the $\overline{\mathbf{N}}$. Compute $\mathrm{Tr}SU(N)^3$.

2. Consider an $SU(N_c)$ gauge theory with a complex matter field Φ^c in the fundamental representation of $SU(N_c)$, $c = 1 \ldots N_c$, with a potential $V = -\frac{1}{2}m^2 \sum_c \Phi^c \Phi^\dagger_c + \frac{\lambda}{4}(\sum_c \Phi^c \Phi^\dagger_c)^2$, leading to $\langle \Phi_c \rangle \neq 0$.

(a) Use the symmetry to argue that you can rotate the vev to point in a particular direction, and use that to determine the unbroken gauge group. Verify that the number of would-be NGBs that are eaten agree with how many of the gauge fields got a mass m_A. Compute the mass m_A.

(b) Suppose that there are N_f flavors of complex matter fields, $\Phi^{c,f}$ with $f = 1 \ldots N_f$, each in the fundamental of the $SU(N_c)$ gauge symmetry. What is the largest possible global flavor symmetry? Suppose that the potential preserves this largest possible symmetry, but leads to a non-zero $\langle \Phi^{c,f} \rangle$ in the vacuum. What is the configuration for $\langle \Phi^{c,f} \rangle$ that preserves the largest gauge and flavor symmetry, and what is this largest possible gauge and flavor symmetry? How many gauge fields get a non-zero mass m_A? How many NGBs were left uneaten, and what is the field-space where they take values?

(c) The theory for $E > m_A$ is the original $SU(N_c)$ theory with the N_f matter fields $\Phi^{f,c}$. For $E < m_A$ we can integrate out the massive fields to get a low-energy theory consisting of the unbroken gauge fields and any uneaten matter. Refer to the lecture on May 11 to write down the one-loop running of the gauge coupling of the theory both above the scale m_A and below the scale m_A. By matching the running coupling, relate the scale Λ_H of the high-energy theory to the scale Λ_L of the low-energy theory. Should we have $m_A > \Lambda_H$ or $m_A < \Lambda_H$ for weak coupling? In that case, is $\Lambda_L > \Lambda_H$ or $\Lambda_L < \Lambda_H$?

3. Please write a one-paragraph outline of the topic that you would like to present for the final presentation. The final will be to write up a short report on the topic (around 5-7 pages), and give a 10 minute presentation during the final exam time slot.