215c Homework exercises 4, Spring 2020, due May 1

1. The $k = 1$ instanton solution of $su(2)$ Yang-Mills was discussed in lecture, and can be found in eqn. 2.51 (p. 53) in Tong's notes.

   (a) Verify that this A_μ leads to the $F_{\mu\nu}$ given on the following line.

   (b) Verify that this leads to $S_{inst} = 8\pi^2/g^2$. Feel free to use $\int d^4x \frac{(x^2)^n}{(x^2+1)^m} = \pi^2 \frac{\Gamma(n+2)\Gamma(m-n-2)}{\Gamma(m)}$.

2. For $x \to \infty$, the $k = 1$ instanton has $A_\mu \to U(i\partial_\mu)U^{-1}$ with $U = x_\mu \sigma^\mu/\sqrt{x^2}$ and $\sigma^\mu = (1, -i\vec{\sigma})$. Define also $\bar{\sigma}^\mu = (1, i\vec{\sigma})$. The $k = -1$ anti-instanton has $A_\mu \to U(i\partial_\mu)U^{-1}$ with $U = x_\mu \bar{\sigma}^\mu/\sqrt{x^2}$. Comparing with the expression for A_μ in terms of $\eta^a_{\mu\nu}\sigma^a$, relate that quantity to σ^μ and $\bar{\sigma}^\mu$. Likewise, for the the anti-instanton, relate its $\bar{\eta}^a_{\mu\nu}\sigma^a$ to σ^μ and σ^ν.

3. Review the chiral spinor notation from my online lecture notes from 215a, on Nov. 20, 2019, e.g. take the basis $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$, and notice that replacing the σ^μ_{there} with the σ^μ_{here} gives γ^μ that satisfies the Euclidean Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$ and $\gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Write the 4-component spinor as $\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$ where $\alpha = 1, 2$ and $\dot{\alpha} = \dot{1}, \dot{2}$. Note that $\not{D}\Psi = \begin{pmatrix} 0 & D_\mu\sigma^\mu \\ D_\mu\bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix}$ and the off-diagonal entries show that $D_\mu\bar{\sigma}_\mu$ acts on a left-handed Fermion to give a right-handed Fermion component, while $D_\mu\sigma_\mu$ acts on a right-handed Fermion component to give a left-handed Fermion component. And if the Fermion is massless, the Dirac equation separates into $\bar{\sigma}_\mu D_\mu\psi = 0$ and $\sigma_\mu D_\mu\bar{\chi} = 0$.

   Recall that $M^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ gives the spinor representation of the Lorentz group. Write out this $M^{\mu\nu}$ in terms of σ^μ and $\bar{\sigma}^\mu$. You should find that it is reducible, corresponding to the fact that the Dirac spinor can be decomposed into chiral and anti-chiral parts. Compare the results with those of the previous question. Using the statement that $\eta^a_{\mu\nu}$ is self-dual and $\bar{\eta}^a_{\mu\nu}$ is anti-self dual, you can infer that the spinor representation of $M_{\mu\nu}$ has decomposed into self-dual and anti-self dual parts. Is the self-dual part associated with left-handed chiral, or right? (Different references use different conventions.)