215c Homework exercises 1, Spring 2020, due April 6

1. The adjoint rep

(a) Using the Jacobi identity, verify that $(T^a)^{bc} = -if^{abc}$ satisfies the Lie algebra's commutation relations. This is the adjoint representation.

(b) Write out adjoint rep for $SU(2)$ and verify that it agrees with the $\langle j=1, m|J^a|j=1, m'\rangle$ for $\hbar = 1$ matrix elements for all $a = 1,2,3$ (recall that $m = \pm 1 \sim (x \pm iy)$.

(c) A general property of the adjoint rep is that its tensor product with any other rep includes that rep. Verify this property for the case of $SU(2)$ (it's enough to quote the known rules for addition of angular momentum).

2. The fundamental rep of $SU(3)$ has generators $T^a = \frac{1}{2}\lambda^a$ where λ^a are the 3×3 Gell-Mann matrices. Please look at the Wikipedia entry for Gell-Mann matrices to see the λ^a, and also the f^{abc} for $SU(3)$.

(a) Think of $(T^a)^{ij} = \langle j|T^a|i\rangle$ with $i = 1,2,3$. The $|i\rangle$ are chosen to be eigenstates of T^3 and T^8. The eigenvalues are called the weights of the representation. For $SU(2)$, the weights are the values of m, running from $-\frac{1}{2}j, \ldots + \frac{1}{2}j$ for the rep labelled by j. For $SU(3)$, the weights are a 2d vector (m_1, m_2), which are the eigenvalues of T^3 and T^8. Plot the weights of the fundamental rep of $SU(3)$, with m_1 on the x axis and m_2 on the y axis.

(b) Under a unitary transformation, the fundamental transforms as $|i\rangle \to U^i{}_j|j\rangle$. Let $|\bar{i}\rangle = |i\rangle^*$, this is called the anti-fundamental rep; note that the anti-fundamental rep thus has generators $T^a_{anti-fund} = -(T^a_{fund})^*$, and that this satisfies the commutation relations. Plot the weights of the anti-fundamental rep of $SU(3)$.

(c) For general $SU(N)$, the tensor product of the fundamental and the anti-fundamental equals the trivial rep plus the adjoint rep. For $SU(2)$ this is the addition of angular momentum formula $(j=\frac{1}{2}) \otimes (j = \frac{1}{2}) = (j=0) \oplus (j=1)$, which we can also write as $2 \times 2 = 1 + 3$. The analog for $SU(N)$ is $N \times \bar{N} = 1 + adj$ where N is the fundamental and $\bar{N}$ is the anti-fundamental. The weights of the tensor product of two reps is the sum of all of the weights (generalizing $m_{tot} = m_1 + m_2$ in addition of angular momentum). Using this, and the results of the previous parts, plot the weights of the adjoint of $SU(3)$.

3. Consider $SU(N)$ and let's introduce the notation that upper indices $i = 1 \ldots N$ refer to the fundamental, and lower indices $i = 1 \ldots N$ denotes the anti-fundamental. So x^i is a fundamental and y_i is an anti-fundamental. An object x^{ij} is in the $N \times N$ tensor product. This rep is reducible because we can impose either $x^{ij} = \pm x^{ji}$, correspondingly $N \times N = \frac{1}{2}N(N+1) + \frac{1}{2}N(N-1)$. For $SU(3)$, this is $3 \times 3 = 6 + \bar{3}$. Plot the weights for both sides of this multiplication rule.