4/25/19 Lecture outline

⋆ Reading: Zwiebach chapters 4,5,6.

• Continue from last time: $S_{p.p.} = -mc\int ds = -mc\int\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$, proportional to the worldline length and reparameterization invariant under $\tau \to \tau'(\tau)$. Likewise, for a string world-sheet, we need two parameters, ξ^a, $a = 1, 2$. The string trajectory is $x : \Sigma \to M$, where Σ is the 2d world-sheet, with local coordinates ξ^a, and M is the target space, with local coordinates x^μ. The worldsheet area element is $A = \int d^2\xi\sqrt{|h|}$, where h_{ab} is the worldsheet metric, and $|h|$ is its determinant. Suppose that the target space has metric $g_{\mu\nu}$, with space-time length e.g. $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. By writing $dx^\mu = \partial_a x^\mu d\xi^a$, we get

$$ds^2 = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b}d\xi^a d\xi^b, \qquad \text{so} \qquad h_{ab} = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b},$$

where this h_{ab} is called the induced metric. So the worldsheet area functional is

$$A = \int d^2\xi\sqrt{\det_{ab}(g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b})}.$$

For strings in Minkowski spacetime, we write it instead as $X^\mu(\tau, \sigma)$. There is also a needed minus sign, as the area element is $\sqrt{|g|}$, actually involves the absolute value of the determinant, and the determinant is negative (just like $\det\eta = -1$). So

$$A = \int d\tau d\sigma\sqrt{(\frac{\partial X}{\partial\tau}\cdot\frac{\partial X}{\partial\sigma})^2 - (\frac{\partial X}{\partial\tau})^2(\frac{\partial X}{\partial\sigma})^2},$$

where the spacetime indices are contracted with the metric $g_{\mu\nu}$. To get an action with $[S] = ML^2/T$, we have

$$S_{Nambu-Goto} = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau\int d\sigma\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2},$$

where we define $\dot{X}^\mu \equiv \frac{dx^\mu}{d\tau}$ and $X^{\mu\prime} \equiv \frac{\partial X^\mu}{\partial\sigma}$ annd T_0 is the string tension, with $[T_0] = [F] = [ML/T^2]$.

The action is reparameterization invariant: can take $(\tau, \sigma) \to (\tau'(\tau, \sigma), \sigma'(\tau, \sigma))$ and get $S \to S$. Enormous symmetry/redundancy in choice of (τ, σ); can "fix the gauge" to some convenient choice, and the physics is completely independent of the choice. This is crucial, since the worldsheet coordinates have no physical significance.

• We can write $S_{NG} = \int d^2\xi\mathcal{L}_{NG}$ with Lagrangian density

$$\mathcal{L}_{NG} = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2},$$

and we have

$$\mathcal{P}^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}},$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}.$$

The condition $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0.$$

For the open string, $\delta S = 0$ also requires $\int d\tau [\delta X^\mu P^\sigma_\mu]_0^{\sigma_0} = 0$, which requires for each μ index either of the Dirichlet or Neumann BCs, at each end:

$$\text{Dirichlet} \qquad \frac{\partial X^\mu}{\partial \tau}(\tau, \sigma_*) = 0 \qquad \to \qquad \delta X^\mu(\tau, \sigma_*) = 0,$$

$$\text{Neumann} \qquad \mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0.$$

• Exploit $(\tau, \sigma) \to (\tau', \sigma')$ reparameterization invariance to pick useful "gauges", to simplify the above equations. We will discuss choices such that we can impose constraints

$$\dot{X}\cdot X' = 0 \qquad \dot{X}^2 + X'^2 = 0. \tag{1}$$

In this case, we have

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu \qquad \mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}X^{\mu'}, \tag{2}$$

and then the EOM is simply a wave equation:

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0. \tag{3}$$

Now let's explain these things in more detail. • We will motivate the above choice by discussing in more detail the interpretation of $X^\mu(\tau, \sigma)$. Consider the tangent vectors $\partial_\tau X^\mu$ and $\partial_\sigma X^\mu$; aside from isolated points, we can and will choose τ and σ such that they are timeline and space-like, respectively. Take $v^\mu(\lambda) = \partial_\tau X^\mu + \lambda \partial_\sigma X^\mu$, so $v^2 = (\dot{X})^2 + 2\lambda \dot{X}\cdot X' + \lambda^2 (X')^2$ which can be either positive or negative, so there must be two real λ solutions to the condition $v^2 = 0$; the condition that this is true is that the descriminant of the quadratic equation must be positive, and that is precisely what is inside the $\sqrt{\cdot}$ in $\mathcal{L}_{NG}$.

Since $\dot{X}^\mu$ is timelike, we can choose static gauge, where $\tau = t$. Verify sign inside $\sqrt{\cdot}$ in this case: $X^{\mu'} = (0, \vec{X}')$, $\dot{X}^\mu = (c, \dot{\vec{X}})$, take e.g. $\dot{\vec{X}} = 0$ to get $\sqrt{\cdot} = c|\vec{X}'|$.

• Consider example of $X^\mu(\sigma, \tau) = (c\tau, f(\sigma), 0, \ldots 0)$. So $\dot{X}^\mu = (c, \vec{0})$ and $X'^\mu = (0, f'(\sigma), 0, \ldots, 0)$. Verify that the EOM are satisfied. Compute the action and note that $V = T_0 a$ where a is the length of the string.