4/23/19 Lecture outline

⋆ Reading: Zwiebach chapters 4,5,6.

• Last time: nonrelativistic strings. $[T_0] = [F] = [E]/L = [\mu_0][v^2]$. Indeed, considering $F = ma$ for an element $dx$ of the string yields the string wave equation $\frac{\partial^2 y}{\partial x^2} - \frac{1}{v_0^2}\frac{\partial^2 y}{\partial t^2} = 0$, with $v_0 = \sqrt{T_0/\mu_0}$. Endpoints at $x = 0$ and $x = a$. Can choose Dirichlet or Neumann BCs at these points. With Dirichlet at each end, $y_n(x) = A_n \sin(n\pi x/a)$ and the general solution is $y(x,t) = \sum_n y_n(x)\cos\omega_n t$, where $\omega_n = v_0 n\pi/a$ (and the $A_n$ are determined from the initial conditions, by Fourier transform).

The nonrelativistic string action is $S = \int dt L$ where $L$ is the kinetic energy minus potential energy, which gives

$$S = \int dt \int dx \left( \tfrac{1}{2}\mu_0 (\frac{\partial y}{\partial t})^2 - \tfrac{1}{2}T_0(\frac{\partial y}{\partial x})^2 \right),$$

which is a particular case of the more general action $S = \int dt dx \mathcal{L}(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x})$. We can then define the momentum density and corresponding spatial quantity

$$\mathcal{P}^t = \frac{\partial \mathcal{L}}{\partial \dot{y}}, \qquad \mathcal{P}^x = \frac{\partial \mathcal{L}}{\partial y'}.$$

The variation of the action is

$$\delta S = \int dt dx [\mathcal{P}^t \delta\dot{y} + \mathcal{P}^x \delta y'] = -\int dt dx [\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x}]\delta y + \text{bndy terms}$$

and the action is made stationary, $\delta S = 0$, if the boundary terms vanish and if

$$\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0,$$

which when applied to the above particular choice of action gives the usual wave equation. The boundary terms must also be set to zero, and they involve $\mathcal{P}^t \delta y$ at the time endpoints and $\mathcal{P}^x \delta y$ at the space endpoints. Neumann BCs is to set $\mathcal{P}^x = 0$ at the spatial endpoints (for all $t$), and Dirichlet BCs is to set $\delta y = 0$ (and thus $\mathcal{P}^t = 0$) at the spatial endpoints.

Summary: string action: $S = \int dt dx \mathcal{L}(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x})$, with momentum densities

$$\mathcal{P}^t = \frac{\partial \mathcal{L}}{\partial \dot{y}}, \qquad \mathcal{P}^x = \frac{\partial \mathcal{L}}{\partial y'}.$$

Least action gives the equations of motion

$$\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0.$$

The non-relativistic string has $\mathcal{L} = \frac{1}{2}\mu_0\dot{y}^2 - \frac{1}{2}T_0y'^2$, which we're going to replace with a relativistic version. For guidance, noted that a relativistic point particle of mass $m$ has $S = -mc\int ds = -mc^2\int dt\sqrt{1-v^2/c^2}$ and noted its reparametrization invariance: write $x_\mu(\tau)$, and can change worldline parameter $\tau$ to an arbitrary new parameterization $\tau'(\tau)$, and the action is invariant. To see this use $S = -mc\int\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$ and change $\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau}$ and note that $S \to S$. Euler Lagrange equations of motion: $\frac{dp_\mu}{d\tau} = 0$.

• As we discussed before, the action for a relativistic point particle of mass m is $S = -mc\int ds = -mc^2\int dt\sqrt{1-v^2/c^2}$. This gives $\vec{p} = \partial_{\vec{v}} = \gamma m\vec{v}$ and $H = \vec{p}\cdot\vec{v} - L = \gamma mc^2$, both of which are constants of the motion (thanks to the time and spatial translation invariance). This has reparametrization invariance: write $x_\mu(\tau)$, and can change worldline parameter $\tau$ to an arbitrary new parameterization $\tau'(\tau)$, and the action is invariant. To see this use $S = -mc\int\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$ and change $\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau}$ and note that $S \to S$. The Euler Lagrange equations of motion are $\frac{dp_\mu}{d\tau} = 0$. When the particle is charged and in the presence of electric and magnetic fields, there is the new term in the action

$$S = \int(-mcds + \frac{q}{c}A_\mu dx^\mu), \tag{1}$$

which is manifestly relativistically invariant (and also reparameterization) invariant. Note also that, under a gauge transformation, we have $S \to S + \frac{qf}{c}$, which does not affect the equations of motion (just as changing the Lagrangian by a total time derivative does not).

The lagrangian is thus $L = -mc\sqrt{1-\vec{v}^2/c^2} + \frac{q}{c}\vec{v}\cdot\vec{A} - q\phi$. The momentum conjugate to $\vec{r}$ is $\vec{P} = \partial L/\partial\vec{v} = m\vec{v}/\sqrt{1-\vec{v}^2/c^2} + \frac{q}{c}\vec{A}$. The Hamiltonian is $H = \vec{v}\cdot\vec{P} - L = \sqrt{m^2c^4 + c^2(\vec{P} - \frac{q}{c}\vec{A})^2} + q\phi$. The equations of motion can be written as $\frac{d^2x^\mu}{d\tau^2} = \frac{q}{mc}F_{\mu\nu}\frac{dx^\nu}{d\tau}$. In the non-relativistic limit we have $H = \frac{1}{2m}(\vec{P} - \frac{q}{c}\vec{A})^2 + q\phi$, where $\vec{P} - \frac{q}{c}\vec{A} = m\vec{v}$.

Recap: $S = -mc\int ds + \frac{q}{c}\int A_\mu dx^\mu$ for a relativistic point particle, where we can write $ds = \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}d\tau$, with $\dot{} \equiv \frac{d}{d\tau}$, and $\tau$ is the arbitrary worldline parameter, with reparameterization symmetry $\tau \to \tau'$.

• For a string world-sheet, we need two parameters, $\xi^a$, $a = 1, 2$. The string trajectory is $x : \Sigma \to M$, where $\Sigma$ is the 2d world-sheet, with local coordinates $\xi^a$, and $M$ is the target space, with local coordinates $x^\mu$. The worldsheet area element is $A = \int d^2\xi\sqrt{|h|}$, where $h_{ab}$ is the worldsheet metric, and $|h|$ is its determinant. Suppose that the target space has metric $g_{\mu\nu}$, with space-time length e.g. $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. By writing $dx^\mu = \partial_a x^\mu d\xi^a$, we get

$$ds^2 = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b}d\xi^a d\xi^b, \qquad \text{so} \qquad h_{ab} = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b},$$

where this $h_{ab}$ is called the induced metric. So the worldsheet area functional is

$$A = \int d^2\xi \sqrt{\det_{ab}(g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b})}.$$