4/18/19 Lecture outline

⋆ Reading: Zwiebach chapters 2 and 3.

• Last time: In 4d, we have gravitational potential given by $\Phi_g^{(4)} = -GM/r$, which solves $\nabla^2\Phi_g^{(D)} = 4\pi G^{(D)}\rho_m$. In any spacetime dimension, take $F_g = -m\nabla\Phi_g$, and $\nabla^2\Phi_g^{(D)} = 4\pi G^{(D)}\rho_m$. So in $\hbar = c = 1$ units, get $[F] = 2$, $[\Phi] = 0$, $[\rho] = D$, so $[G] = 2-D$. Take $G = \ell_P^{D-2}$ in D spacetime dimensions.

Get $G^D = GV_C$, where V_C is the compactification volume. Example: consider a string of tension T that is wrapped on a circle in a 5th dimension, of radius ℓ_C. So $\rho_{5d} = T\delta(x_1)\delta(x_2)\delta(x_3)$ and $\rho_{4d} = \ell\rho_{5d}$, i.e. the 4d mass is $M = T\ell_c$, and the potential is $\Phi = -G^4M/r = -G^4T\ell_C/r$. Fits with $G^5 = G^4\ell_C$.

More generally, if we dimensionally reduce, then $G^{-1}_{reduced} = G^{-1}_{original}V_C$.

• Gravity is described, according to Einstein, by taking $ds^2 = g_{\mu\nu}dx^\mu dx^\mu$, with $g_{\mu\nu}$ dynamical and analogous to A_μ. The analog of $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the Einstein Hilbert action, $S \supset \int d^Dx\sqrt{|g|}\frac{1}{16\pi G}R$, where R is called the Ricci scalar curvature. We will not discuss it in detail, and the only point that I'd like to make for the moment is that it is convenient to take mass dimensions such that $[x] = -1$ and $[g_{\mu\nu}] = 0$ and then, since R is built from second derivatives of $g_{\mu\nu}$, it has $[R] = 2$, so $[G^{-1}] = D-2$, i.e. $G \sim \ell_P^{D-2} \sim M_P^{2-D}$. This fits with the force between two masses being $F \sim Gm_1m_2/r^{D-2}$.

If we dimensionally reduce, get $G^{-1}_{reduced} = G^{-1}_{original}V$, as above.

• The electric and magnetic fields themselves have a lagrangian, with action

$$S = \int d^Dx\mathcal{L}, \qquad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{c}A_\mu j^\mu.$$

The two Maxwell's equations expressing absence of magnetic monopoles are, again, solved by setting $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$. The other two Maxwell's equations then come from the Euler -Lagrange equations of the above action upon varying $A_\mu \to A_\mu + \delta A_\mu$: the action is stationary when

$$\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)} - \frac{\partial\mathcal{L}}{\partial A_\mu} = 0.$$

It is convenient to rescale A_μ and j^μ such that the unit of electric charge is 1 instead of the charge e of an electron. Also, I will use g instead of e. Doing so, g only appears in the kinetic terms for the gauge fields: $\mathcal{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}$. This applies in any D, and in any D, the mass dimensions are $[F_{\mu\nu}] = 2$ and $[\mathcal{L}] = D$, so $[g^{-2}] = D-4$. The force between two point charges separated by distance r is $\sim g^2r^{1-d}$ and $[F] = [ma] = 2 = 4-D+d-1$ checks.

If we dimensionally reduce, then $g_{reduced}^{-2} = g_{original}^{-2} V$.

• Nonrelativistic strings. $[T_0] = [F] = [E]/L = [\mu_0][v^2]$. Indeed, considering $F = ma$ for an element dx of the string yields the string wave equation $\frac{\partial^2 y}{\partial x^2} - \frac{1}{v_0^2}\frac{\partial^2 y}{\partial t^2} = 0$, with $v_0 = \sqrt{T_0/\mu_0}$. Endpoints at $x = 0$ and $x = a$. Can choose Dirichlet or Neumann BCs at these points. With Dirichlet at each end, $y_n(x) = A_n \sin(n\pi x/a)$ and the general solution is $y(x,t) = \sum_n y_n(x) \cos \omega_n t$, where $\omega_n = v_0 n\pi/a$ (and the A_n are determined from the initial conditions, by Fourier transform).