6/6/19 Lecture outline

$\star$ Reading: Zwiebach chapter 13

• Last time: we saw that open strings have $M^2 = \alpha'^{-1}(-1+N^\perp)$ and closed strings have $M^2 = \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2)$. For open string for example states are $|\lambda\rangle = \prod_{n=1}^\infty \prod_{I=2}^{25} (a_n^{I\dagger})^{\lambda_{n,I}} |p^+, \vec{p}_T\rangle$ and $N^\perp = \sum_I \sum_n n\lambda_{n,I}$. The number of states with a given value of $N^\perp$ is related to the partitions of $N^\perp$ and this is generated by $f(x) = \mathrm{Tr}_{states} x^{\alpha' M^2}$. Find

$$f_{os}(x) = x^{-1} \prod_{n=1}^\infty \frac{1}{(1-x^n)^{24}} = \frac{1}{\eta^{24}}$$

where we set $D-2 = 24$. Similarly, for the closed string case, we have $f(x) = \mathrm{Tr}_{states} x^{\frac{1}{2}\alpha' M^2}$.

$$f_{closed}(x, \bar{x}) = f_{os}(x) f_{os}(\bar{x}),$$

where we need to project out those states with different powers of x and $\bar{x}$. The function η is Dedekind's eta function,

$$\eta(\tau) \equiv e^{i\pi\tau/12} \prod_{n=1}^\infty (1 - e^{2\pi i n\tau})$$

and we set $x = e^{2\pi i\tau}$. The eta function was defined long ago by mathematicians. It connects to number theory via partitions of N and τ is associated with the complex structure of a torus, given by $z \cong z+1 \cong z+\tau$, with $\tau \cong \tau + 1 \cong -1/\tau$.

• The number of string states grows very rapidly with excitation number, and it turns out that this puts an upper limit on the temperature, beyond which the partition function would not converge. Write

$$f_{os}(x) = \mathrm{Tr}_{states} x^{\alpha' M^2} = \left(\frac{1}{x^{1/24}\prod_{n=1}^\infty (1-x^n)}\right)^{(D-2)} \equiv \sum_{N=0}^\infty p_{D-2}(N) x^{N-(D-2)/24}.$$

We saw that $D=26$, but let's keep it as a parameter for the moment. Here $p_{D-2}(N)$ is the number of distinct partitions of N into arbitrary numbers of non-negative integers, each of which can have $D-2$ labels. This corresponds to how many choices of $\lambda_{I,n}$ there are such that $\prod_{I=2}^{D-1}\prod_{n=1}^\infty (a_n^{I\dagger})^{\lambda_{I,n}}$ has $N^\perp = \sum_n \sum_I n\lambda_{I,n} = N$. Let's consider $p_1(N) = p(N)$ as an illustration: $p(5) = 7$, $p(10) = 42$, find $p(N)$ grows rapidly with N. The large N behavior of $p(N)$ was studied long ago by number theorists Hardy and Ramanujan:

$$p(N \gg 1) \approx \frac{1}{4N\sqrt{3}} \exp(2\pi\sqrt{\frac{N}{6}}).$$

Can also show:

$$p_b(N \gg 1) \approx \frac{1}{\sqrt{2}} \left(\frac{b}{24}\right)^{(b+1)/2} N^{-(b+3)/4} \exp(2\pi\sqrt{\frac{Nb}{6}}).$$

Note the appearance of 24 in this number theory formula, which will be related to $D-2 = 24$ in string theory. In fact, these formula were derived by relating the above generating functions to the Dedekind eta function:

$$\eta(\tau) \equiv e^{i\pi\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}).$$

An important property of the eta function (both for math, and for string theory!) is

$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau),$$

and this allows us to relate the $x \to 1$ limit, which is relevant for extracting $p(N \gg 1)$, to another limit: $x \to 1$ is $\tau \to 0i$, which can be related to $-1/\tau \to i\infty$.

Setting $\tau = i\tau_2$, the above generating functions start to resemble partition functions, with $H = \alpha' M^2$. (Actually, they are partition functions, but on the worldsheet for the moment.)

$$f_{os} = \mathrm{Tr}_{states} e^{-2\pi\tau_2\alpha' M^2} = \sum_{N=0}^{\infty} p_{D-2}(N) e^{-2\pi\tau_2(N-1)}.$$

For large N, we have $M^2 \approx N/\alpha'$ and, taking $E = M$, we have $\sqrt{N} = \sqrt{\alpha'}E$. The entropy of string states with energy E is then

$$S(E) = k \ln \Omega(E) = k \ln p_{24}(N = \sqrt{\alpha' E}) \approx k 4\pi\sqrt{\alpha'}E.$$

Then

$$\frac{1}{kT} = \frac{1}{k}\frac{\partial S}{\partial E} = 4\pi\sqrt{\alpha'} \equiv \frac{1}{kT_H},$$

where T_H is the Hagedorn temperature. For $T \to T_H$ the partition function diverges, so either T_H is the hottest temperature or there must be a phase transition there, with something else happening at higher temperatures (e.g. for the QCD temperature this could be deconfinement into quarks and gluons).

• Consider the closed, bosonic string on a circle, $X_{25} \sim X_{25} + 2\pi R$. If we were dealing with particles rather than strings, we know what would happen: the momentum in the circle direction is quantized (by $\psi \sim e^{ip\cdot x}$ being set equal to itself when going around the circle) as

$$p_{25} = \frac{n}{R}, \qquad n = 0, \pm 1, \pm 2 \ldots.$$

For a big circle, these are closely spaced together, and for a small circle they are widely separated. That's why it's hard to experimentally rule out the absence of tiny, rolled up, extra dimensions: it could just take more energy than we can make presently to excite one of the $n \neq 0$ "Kaluza-Klein modes."

Now we're going to describe something bizarre about strings: there is a symmetry, called T-dualtiy, which makes the physics invariant under $R \leftrightarrow \alpha'/R$. This is strange: a very big circle is physically indistinguishable from a very small circle! The reason is that, in addition to momentum, there are string winding modes, and T-duality exchanges them. For a big circle, the momentum modes are light and the winding modes are heavy, and for a tiny circle they're reversed, but same physics. Smallest possible effective distance, $R = \sqrt{\alpha'}$.

The winding number is given by $X(\tau, \sigma + 2\pi) - X(\tau, \sigma) = m(2\pi R)$. We then have $X = X_L + X_R$ with

$$X_L(\tau + \sigma) = const. + \tfrac{1}{2}\alpha'(p + w)(\tau + \sigma) + \text{oscillators},$$

$$X_R(\tau - \sigma) = const + \tfrac{1}{2}\alpha'(p - w)(\tau - \sigma) + \text{oscillators}.$$

Here

$$p = \frac{n}{R}, \qquad w = \frac{mR}{\alpha'}.$$

The T-duality symmetry comes from the symmetry $(p_L, p_R) \to (p_L, -p_R)$, where

$$p_L = \frac{n}{R} + \frac{mR}{\alpha'}, \qquad p_R = \frac{n}{R} - \frac{mR}{\alpha'}.$$

Also, to have $X(\tau, \sigma + 2\pi) \sim X(\tau, \sigma) + 2\pi R m$, we need $N^\perp - \widetilde{N}^\perp = nm$.

• Now consider the superstrings. The bosonic string has fields $X^I(\tau, \sigma)$, which are $D - 2$ worldsheet scalars. Now we introduce $D - 2$ worldsheet fermions

$$\Psi_R(\tau - \sigma)^I, \qquad \Psi_L^I(\tau + \sigma).$$

Here R and L are for right and left moving, and $I = 2 \ldots D - 2$ (spacetime vector indices). There are two choices of boundary conditions for left movers, and similarly two choices for right movers:

$$\Psi^I(\tau, \sigma + 2\pi) = \pm \Psi^I(\tau, \sigma), \qquad + : \text{Ramond}, \quad - : \text{Nevu-Schwarz}.$$

In the NS sector we have

$$\Psi^I_{NS} \sim \sum_{n=-\infty}^{\infty} b^I_{n+\frac{1}{2}} e^{-i(n+\frac{1}{2})(\tau-\sigma)}.$$

In the R sector we have

$$\Psi^I_R \sim \sum_{n=-\infty}^{\infty} d^P_n e^{-in(\tau-\sigma)}.$$

The modes satisfy

$$\{b^I_r, b^J_s\} = \delta_{r+s,0}\delta^{IJ}, \qquad \{d^I_n, d^J_m\} = \delta_{n+m,0}\delta^{IJ},$$

where $\{A, B\} \equiv AB + BA$ is the anti-commutator, reflecting the fermionic nature of the modes.

The NS sector states are

$$|\lambda, \rho\rangle_{NS} = \prod_{I=2}^{D-2} (a^{I\dagger}_n)^{\lambda_{n,I}} \prod_{J=2}^{D-1} \prod_{r=\frac{1}{2},\frac{3}{2}\cdots} (b^J_{-r})^{\rho_{r,J}} |NS\rangle \otimes |p\rangle,$$

where the $\rho_{r,J}$ are either zero or one (Fermi statistics).

The R sector states are

$$|\lambda, \rho\rangle_R = \prod_{I=2}^{D-2} \prod_n (a^{I\dagger}_n)^{\lambda_{n,I}} \prod_{J=2}^{D-1} \prod_{m=1}^{\infty} (d^J_{-m})^{\rho_{m,J}} |R_A\rangle \otimes |p\rangle,$$

Here $|R_A\rangle$ are the Ramond ground states, which are complicated thanks to the zero modes d^I_0. We take half of them $\frac{1}{2}(D-2)$ to be creation operators and the other half to annihilate the vacuum. So then there are $2^{\frac{1}{2}(D-2)}$ degenerate states. These form two equal groups, depending on whether there are an even number of creation operators, or an odd number. The former is called the $R-$ sector and labeled by $|R_a\rangle$, and the latter is called the $R+$ sector and labeled by $|R_{\bar{a}}\rangle$. The $\pm$ refer to worldsheet fermion number $(-1)^F$, where the vacuum has fermion number $(-1)^F = -1$.

- The mass-squared operator in the NS sector before normal ordering is

$$\alpha' M^2 = \tfrac{1}{2} \sum_{p\neq 0} \alpha^I_{-p}\alpha^I_p + \tfrac{1}{2} \sum_{r=n+\frac{1}{2}} r b^I_{-r} b^I_r.$$

Re-ordering, we have

$$\alpha' M^2 = N^{\perp} + \tfrac{1}{2}(D-2)(-\frac{1}{12} - \frac{1}{24}),$$

where the $-1/12$ was seen in the bosonic case, and the $-1/24$ is the analog coming from reordering the b_r. As in the bosonic case, the commutator $[M^{-I}, M^{-J}] = 0$ determines the spacetime dimension, here to be $D = 10$. So in the NS sector the mass squared operator is

$$\alpha' M^2 = -\tfrac{1}{2} + N^\perp, \qquad N^\perp = \sum_{p=1}^{\infty} p a_p^{\dagger I} a_p^I + \sum_{r=\frac{1}{2},\frac{3}{2},\ldots} r b_{-r}^I b_r^I.$$

Similarly, in the R-sector, we have

$$\alpha' M^2 = \tfrac{1}{2} \sum_{p \neq 0} \alpha_{-p}^I \alpha_p^I + \tfrac{1}{2} \sum_m m d_{-m}^I d_m^I.$$

Re-ordering we have $\alpha' M^2 = N^\perp + \frac{1}{2}(D-2)(-\frac{1}{12} + \frac{1}{12})$, and the constants cancel, so

$$\alpha' M^2 = N^\perp, \qquad N^\perp = \sum_{p=1}^{\infty} p a_p^{\dagger I} a_p^I + \sum_{m=1}^{\infty} m d_{-m}^I d_m^I.$$

In particular, the Ramond ground states are massless.

- The NS spectrum generating function is

$$f_{NS}(x) = \frac{1}{\sqrt{x}} \prod_{n=1}^{\infty} \left(\frac{1 + x^{n-\frac{1}{2}}}{1 - x^n} \right)^8 .$$

The R sector spectrum generating function is

$$f_{R\pm}(x) = 8 \prod_{n=1}^{\infty} \left(\frac{1 + x^n}{1 - x^n} \right)^8$$

where 8 accounts for the ground state degeneracy associated with d_0^I, in either the R_+ or the R_- sector. We should also GSO project the NS sector, i.e. throw away states with $(-1)^F = -1$ to get the $NS+$ states, with generating function

$$f_{NS+}(x) = \frac{1}{2\sqrt{x}} \left[\prod_{n=1}^{\infty} \left(\frac{1 + x^{n-\frac{1}{2}}}{1 - x^n} \right)^8 - \left(\frac{1 - x^{n-\frac{1}{2}}}{1 - x^n} \right)^8 \right].$$

This projects out the tachyon – nice! Moreover, the states in $f_{R\pm}$ are spacetime fermions, whereas those in $f_{NS,+}$ are spacetime bosons, and their spectrum is degenerate, thanks to the identity $f_{R\pm}(x) = f_{NS+}(x)$ (which was proven as a mathematical identity around 150 years before the superstring was even first thought of!).

• For closed superstrings we can take the $NS+$ sector for both left and right movers, and the $R-$ sector for both left and right movers; this is the IIB superstring. Or we could take the $NS+$ sector for both left and right movers, and the $R-$ sector for left movers and the $R+$ sector for right movers; this is the IIA superstring.

The massless (NS+, NS+) states for both of these string theories consist of

$$\widetilde{b}^I_{-\frac{1}{2}}|NS\rangle_L \otimes b^J_{-\frac{1}{2}}|NS\rangle_R \otimes |p\rangle.$$

As in the bosonic case, these correspond to $g_{\mu\nu}$,, $B_{\mu\nu}$, ϕ.