5/30/19 Lecture outline

⋆ Reading: Zwiebach chapter 12

• Recall from last time: relativistic NG open string.

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}X^{\mu'}, \qquad \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^{\mu}.$$

$$X^I(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I\cos n\sigma e^{-in\tau}. \tag{1}$$

where $\alpha_{-n}^{\mu} \equiv \alpha_n^{\mu *}$ (to make X^μ real) and it's also convenient to define $\alpha_0^\mu \equiv \sqrt{2\alpha'}p^\mu$. Then

$$\dot{X}^\mu \pm X^{\mu'} = \sqrt{2\alpha'}\sum_{n=-\infty}^{\infty}\alpha_n^\mu e^{-in(\tau\pm\sigma)}.$$

$X^+ = \beta\alpha' p^+\tau$ and $p^+ = 2\pi\mathcal{P}^{\tau +}/\beta$ (again, $\beta = 2$ for open strings and $\beta = 1$ for closed strings). The constraints gave

$$\sqrt{2\alpha'}\alpha_n^- \equiv \frac{1}{p^+}L_n^\perp, \qquad L_n^\perp = \tfrac{1}{2}\sum_{m=-\infty}^{\infty}\alpha_{n-m}^I\alpha_m^I.$$

$$[X^I(\sigma), \mathcal{P}^{\tau J}(\sigma')] = i\eta^{IJ}\delta(\sigma-\sigma'), \qquad [x_0^-, p^+] = -i.$$

$$H = 2\alpha' p^+ p^- = 2\alpha' p^+\int_0^\pi d\sigma\mathcal{P}^{\tau -} = \pi\alpha'\int_0^\pi d\sigma(\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + X^{I'}X^{I'}(2\pi\alpha')^{-2}) = L_0^\perp$$

$$[\alpha_m^I, \alpha_n^J] = m\eta^{IJ}\delta_{n+m,0}.$$

Summary: we fix X^+ to be simply related to τ, find that the X^I are given by simple harmonic oscillators, and X^- is a complicated expression, fully determined in terms of the transverse direction quantities:

$X^+(\tau,\sigma) = 2\alpha' p^+\tau = \sqrt{2\alpha'}\alpha_0^+\tau$. For X^- recall expansion, with $\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n^\perp$, where $L_n^\perp \equiv \frac{1}{2}\sum_p \alpha_{n-p}^I\alpha_p^I$ is the transverse Virasoro operator. Recall $[\alpha_m^I, \alpha_n^J] = m\delta^{IJ}\delta_{m+n,0}$. There is an ordering ambiguity here, only for $L_0^\perp$:

$$L_0^\perp = \tfrac{1}{2}\alpha_0\alpha_0 + \tfrac{1}{2}\sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I + \tfrac{1}{2}\sum_{p=1}^{\infty}\alpha_p^I\alpha_{-p}^I.$$

The ordering in the last terms need to be fixed, so the annihilation operator α_p is on the right, using $\alpha_p^I\alpha_{-p}^I = \alpha_{-p}^I\alpha_p^I + [\alpha_p^I, \alpha_{-p}^I]$, which gives

$$L_0^\perp = \alpha' p^I p^I + \sum_{n=1}^{\infty} n a_n^{I\dagger}a_n^I,$$

where the normal ordering constant has been put into

$$2\alpha' p^- = \frac{1}{p^+}(L_0^\perp + a), \qquad a = \tfrac{1}{2}(D-2)\sum_{p=1}^{\infty} p.$$

This leads to

$$M^2 = \frac{1}{\alpha'}(a + \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I).$$

The divergent sum for a is regulated by using $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and analytically continuing to get $\zeta(-1) = -1/12$. So

$$a = -\frac{1}{24}(D-2).$$

• Virasoro generators and algebra (corresponds to worldsheet energy-momentum tensor). Since $(\alpha_n^I)^\dagger = \alpha_{-n}^I$, get $L_n^{\perp\dagger} = L_{-n}^\perp$. Also,

$$[L_m^\perp, \alpha_n^I] = -n\alpha_{n+m}^I.$$

$$[L_m^\perp, L_n^\perp] = (m-n)L_{n+m}^\perp + \frac{D-2}{12}(m^3 - m)\delta_{m+n,0}.$$

• Spacetime Lorentz symmetry corresponds to conserved currents on worldsheet, with conserved charges

$$M_{\mu\nu} = \int_0^\pi (X_\mu \mathcal{P}_\nu^\tau - (\mu \leftrightarrow \nu))d\sigma.$$

Plug in $\mathcal{P}_\nu^\tau = \frac{1}{2\alpha'}\dot{X}^\mu$ and plug in oscillator expansion of X^μ to get

$$M^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu).$$

In light cone gauge, have to be careful with M^{-I}, since X^- is constrained to something complicated,

$$X^-(\tau,\sigma) = x_0^- + \sqrt{2\alpha'}\alpha_0^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\tau}\cos n\sigma, \qquad \sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n^\perp.$$

and also careful to ensure that $[M^{-I}, M^{-J}] = 0$. Find, after appropriately ordering terms,

$$M^{-I} = x_0^- p^I - \frac{1}{4\alpha' p^+}\left(x_0^I(L_0^\perp + a) + (L_0^\perp + a)x_0^I\right) - \frac{i}{\sqrt{2\alpha'}p^+}\sum_{n=1}^{\infty}\frac{1}{n}(L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp).$$

Then get

$$[M^{-I}, M^{-J}] = -\frac{1}{\alpha' p^{+2}} \sum_{m=1}^{\infty} (\alpha^I_{-m}\alpha^J_m - (I \leftrightarrow J))[m(1-((D-2)/24)) + m^{-1}(((D-2)/24)+a)].$$

Since this must be zero, get $D = 26$ and $a = -1$.

The worldsheet Hamiltonian is thus

$$H = 2\alpha' p^+ p^- = L_0^\perp - 1.$$

- The states are obtained as

$$|\lambda\rangle = \prod_{n=1}^{\infty} \prod_{I=2}^{25} (a_n^{I\dagger})^{\lambda_{n,I}} |p^+, \vec{p}_T\rangle.$$

These states are eigenstates of

$$M^2 = \frac{1}{\alpha'}(-1 + N^\perp), \qquad N^\perp \equiv \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I,$$

with eigenvalues

$$M^2 = \frac{1}{\alpha'}(-1 + N^\perp), \qquad N^\perp = \sum_n \sum_I n\lambda_{n,I}.$$

The groundstate is tachyonic (!). The first excited state is a massless spacetime vector with $D-2$ polarizations, i.e. a massless gauge field, like the photon (but in $D = 26$)!

The tachyon is related to the fact that the D25 brane is unstable, it decays to the closed string vacuum. The closed bosonic string is also unstable, as we'll see next time. These instabilities can be cured by adding fermions and considering the superstring. Then the critical spacetime dimension is $D = 10$.

The eigenstates satisfy the worldsheet SE:

$$i\frac{\partial}{\partial \tau}|\lambda\rangle = H|\lambda\rangle = (L_0^\perp - 1)|\lambda\rangle.$$

Writing $x^+ = 2\alpha' p^+ \tau$, this becomes

$$(i\frac{\partial}{\partial x^+} - \frac{1}{2p^+}(p^I p^I + M^2))\phi_\lambda(x^+, p^+, \tau),$$

which is the KG (or generalization) wave equation for the corresponding field in spacetime.

- Now consider closed string case. Recall gauge conditions $n \cdot X = \alpha'(n \cdot p)\tau$, $n \cdot p = 2\pi n \cdot \mathcal{P}^\tau$, which yielded the constraints $(\dot{X} \pm X')^2 = 0$ and then the EOM were simply $(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0$. For the closed string, this means that $X^\mu(\tau,\sigma) = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma)$. The general solutions can then be written as

$$X_R^\mu(v) = \tfrac{1}{2}x_0^\mu + \sqrt{\tfrac{1}{2}\alpha'}\alpha_0^\mu v + i\sqrt{\tfrac{1}{2}\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-inv},$$

and a similar expression for X_L^μ, with modes $\widetilde{\alpha}_n^\mu$. Since $X^\mu(\tau, \sigma+2\pi) = X^\mu(\tau,\sigma)$, $\widetilde{\alpha}_0^\mu = \alpha_0^\mu$. Computing $\mathcal{P}^{\mu\mu} = \dot{X}^\mu/2\pi\alpha'$ then yields $\alpha_0^\mu = \sqrt{\tfrac{1}{2}\alpha'}p^\mu$.