5/28/19 Lecture outline

⋆ Reading: Zwiebach chapters 11, 12

• Recall from last time: quantize the point particle in light cone gauge by taking the independent operators $(x^I, x_0^-, p^I, p^+)$, with $[x^I, p^J] = i\eta^{IJ}$ and $[x_0^-, p^+] = i\eta^{-+} = -i$. These commutators are for either S or H picture, with the operators being functions of $\tau$ in the H picture. The remaining variables are defined by $x^+(\tau) = p^+\tau/m^2$, $x^-(\tau) = x_0^- + p^-\tau/m^2$, $p^- = (p^I p^I + m^2)/2p^+$ (the first two are explicitly $\tau$ dependent even in the S picture). The Hamiltonian is $\sim p^-$, which generates $\frac{\partial}{\partial x^+}$ translations. Since $\frac{\partial}{\partial \tau} = \frac{p^+}{m^2}\frac{\partial}{\partial x^+} \leftrightarrow \frac{p^+}{m^2}p^-$ the Hamiltonian is

$$H = \frac{p^+p^-}{m^2} = \frac{1}{2m^2}(p^I p^I + m^2).$$

Verify e.g.

$$i\frac{d}{d\tau}p^\mu = [p^\mu, H] = 0, \qquad i\frac{dx^I}{d\tau} = [x^I, H] = i\frac{p^I}{m^2},$$

reproducing the correct EOM. Likewise, verify $\dot{x}_0^- = 0$ and $\dot{x}^+ = \partial_\tau x^+ = p^+/m^2$. The momentum eigenstates are labeled by $|p^+, p^I\rangle$ and these are also energy eigenstates, $H|p^+, p^I\rangle = \frac{1}{2m^2}(p^I p^I + m^2)|p^+, p^I\rangle$.

Connect the quantized point particle with the excitations of scalar field theory via

$$|p^+, p^I\rangle \leftrightarrow a^\dagger_{p^+, p^I}|\Omega\rangle.$$

The S.E. of the quantum point particle wavefunction maps to the classical scalar field equations, e.g. in light cone gauge:

$$(i\partial_\tau - \frac{1}{2m^2}(p^I p^I + m^2))\phi(\tau, p^+, p^I) = 0$$

is either the quantum S.E. of the point particle or the classical field equations of a scalar field.

(Aside: the light cone is here used as a trick to get to "second quantization." "First quantization" is what you learn the first time you study (non-relativisitic) QM: replace coordinates and momenta with operators, and Poisson brackets with commutators. Second quantization is for field theory, replacing the fields and their conjugate momenta with operators, and their PBs with commutators, leading to multi-particle states. Here light-cone first quantization of the point particle leads to a Schrodinger equation that agrees with the classical EOM of a light-cone field theory, which we then need to quantize again to get second quantization.

• We now repeat similar steps for the relativistic NG open string. Recall that we imposed constraints $(\dot{X} \pm X')^2 = 0$ to get

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X^{\mu'}, \qquad \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^{\mu}.$$

The solution of the string EOM (with Neumann BCs at the ends) is then

$$X^I(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\alpha_0^I \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^I \cos n\sigma e^{-in\tau}. \tag{1}$$

where $\alpha_{-n}^\mu \equiv \alpha_n^{\mu *}$ (to make $X^\mu$ real) and it's also convenient to define $\alpha_0^\mu \equiv \sqrt{2\alpha'} p^\mu$. Then

$$\dot{X}^\mu \pm X^{\mu'} = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in(\tau\pm\sigma)}.$$

Recall that in light cone gauge we had $X^+ = \beta\alpha' p^+ \tau$ and $p^+ = 2\pi \mathcal{P}^{\tau +}/\beta$ (again, $\beta = 2$ for open strings and $\beta = 1$ for closed strings). The constraints gave

$$\sqrt{2\alpha'}\alpha_n^- \equiv \frac{1}{p^+} L_n^\perp, \qquad L_n^\perp = \tfrac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m}^I \alpha_m^I.$$

In light cone gauge, much as with the point particle, the independent variables are $(X^I(\sigma), x_0^-, \mathcal{P}^{\tau I}(\sigma), p^+)$. In the H picture the capitalized ones depend (implicitly) on $\tau$ too. The commutation relations are

$$[X^I(\sigma), \mathcal{P}^{\tau J}(\sigma')] = i\eta^{IJ}\delta(\sigma - \sigma'), \qquad [x_0^-, p^+] = -i.$$

The Hamiltonian is taken to be

$$H = 2\alpha' p^+ p^- = 2\alpha' p^+ \int_0^\pi d\sigma \mathcal{P}^{\tau -} = \pi\alpha' \int_0^\pi d\sigma (\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + X^{I'} X^{I'} (2\pi\alpha')^{-2})$$

Can write $H = L_0^\perp$ since $L_0^\perp = 2\alpha' p^+ p^-$. This $H$ properly yields the expected time derivatives, e.g. $\dot{X}^I = 2\pi\alpha' \mathcal{P}^{\tau I}$.

In terms of (1), the needed commutators are ensured by

$$[\alpha_m^I, \alpha_n^J] = m\eta^{IJ}\delta_{n+m,0}.$$

Also, as before, we define $\alpha_0^I \equiv \sqrt{2\alpha'}p^I$. Now define $\alpha^\mu_{n>0} = \sqrt{n}a^\mu_n$ and $\alpha^\mu_{-n} = a^{\mu *}_n\sqrt{n}$ to rewrite the above as

$$[a^I_m, a^{J^\dagger}_n] = \delta_{m,n}\eta^{IJ}. \tag{2}$$

- The transverse light cone coordinates can be described by

$$S_{l.c.} = \int d\tau d\sigma \frac{1}{4\pi\alpha'}(\dot{X}^I\dot{X}^I - X^{I'}X^{I'}).$$

Gives correct $\mathcal{P}^{\tau I} = \partial\mathcal{L}/\partial\dot{X}^I$ and correct $H = \int d\sigma(\mathcal{P}^{\tau I}\dot{X}^I - \mathcal{L})$.

Writing $X^I(\tau,\sigma) = q^I(\tau) + 2\sqrt{\alpha'}\sum_{n=1}^\infty q^I_N(\tau)n^{-1/2}\cos n\sigma$ and plugging into the action above gives

$$S = \int d\tau[\frac{1}{4\alpha'}\dot{q}^I\dot{q}^I + \sum_{n=1}^\infty(\frac{1}{2n}\dot{q}^I_n\dot{q}^I_n - \frac{n}{2}q^I_n q^I_n)]$$

and

$$H = \alpha' p^I p^I + \sum_{n=1}^\infty \frac{n}{2}(p^I_n p^I_n + q^I_n q^I_m).$$

A bunch of harmonic oscillators. Relate to (1) and (2), showing that the $a_m$ can be interpreted as the usual harmonic oscillator annihilation operators.

- Summary: we fix $X^+$ to be simply related to $\tau$, find that the $X^I$ are given by simple harmonic oscillators, and $X^-$ is a complicated expression, fully determined in terms of the transverse direction quantities:

$X^+(\tau,\sigma) = 2\alpha' p^+\tau = \sqrt{2\alpha'}\alpha_0^+\tau$. For $X^-$ recall expansion, with $\sqrt{2\alpha'}\alpha^-_n = \frac{1}{p^+}L^\perp_n$, where $L^\perp_n \equiv \frac{1}{2}\sum_p \alpha^I_{n-p}\alpha^I_p$ is the transverse Virasoro operator. Recall $[\alpha^I_m, \alpha^J_n] = m\delta^{IJ}\delta_{m+n,0}$. There is an ordering ambiguity here, only for $L^\perp_0$:

$$L^\perp_0 = \tfrac{1}{2}\alpha_0\alpha_0 + \tfrac{1}{2}\sum_{p=1}^\infty \alpha^I_{-p}\alpha^I_p + \tfrac{1}{2}\sum_{p=1}^\infty \alpha^I_p\alpha^I_{-p}.$$

The ordering in the last terms need to be fixed, so the annihilation operator $\alpha_p$ is on the right, using $\alpha^I_p\alpha^I_{-p} = \alpha^I_{-p}\alpha^I_p + [\alpha^I_p, \alpha^I_{-p}]$, which gives

$$L^\perp_0 = \alpha' p^I p^I + \sum_{n=1}^\infty n a^{I\dagger}_n a^I_n,$$

where the normal ordering constant has been put into

$$2\alpha' p^- = \frac{1}{p^+}(L^\perp_0 + a), \qquad a = \tfrac{1}{2}(D-2)\sum_{p=1}^\infty p.$$

This leads to

$$M^2 = \frac{1}{\alpha'}(a + \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I).$$

The divergent sum for $a$ is regulated by using $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and analytically continuing to get $\zeta(-1) = -1/12$. So

$$a = -\frac{1}{24}(D-2).$$