5/21/19 Lecture outline

⋆ Reading: Zwiebach chapter 10.

Recall from last time: consider quantization of fields and then strings. As a warmup, consider classical scalar field theory, with $S = \int d^D x(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2)$. The EOM is the Klein-Gordon equation

$$(\partial^2 - m^2)\phi = 0, \qquad \partial^2 \equiv -\frac{\partial^2}{\partial t^2} + \nabla^2$$

The Hamiltonian is $H = \int d^{D-1}x(\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2)$, where $\Pi = \partial\mathcal{L}/\partial(\partial_0\phi) = \partial_0\phi$. Take e.g. $D = 1$ and get SHO with $q \to \phi$ and $m \to 1$ and $\omega \to m$.

Classical plane wave solutions: $\phi(t, \vec{x}) = ae^{-iEt+i\vec{p}\cdot\vec{x}} + c.c.$, where $E = E_p = \sqrt{\vec{p}^2 + m^2}$, and the $+c.c.$ is to make $\phi$ real. Letting $\phi(x) = \int \frac{d^D p}{(2\pi)^D} e^{ip\cdot x}\phi(p)$, the reality condition is $\phi(p)^* = \phi(-p)$ and the EOM is $(p^2 + m^2)\phi(p) = 0$.

• Now consider light cone gauge coordinates. Replace $\partial^2 \to -2\partial_+\partial_- + \partial_I\partial_I$ and Fourier transform

$$\phi(x^+, x^-, \vec{x}_T) = \int \frac{dp^+}{2\pi} \int \frac{d^{D-2}\vec{p}_T}{(2\pi)^{D-2}} e^{-ix^-p^+ + i\vec{x}_T\cdot\vec{p}_T}\phi(x^+, p^+, \vec{p}_T).$$

Then the EOM becomes

$$(i\frac{\partial}{\partial x^+} - \frac{1}{2p^+}(p^I p^I + m^2))\phi(x^+, p^+, \vec{p}_T) = 0.$$

Looks like the non-relativistic Schrodinger equation, with $x_+$ playing the role of time and $p^+$ playing the role of mass, even though it is secretly relativistic.

• Let's quantize! Replace $\phi$ with an operator. Consider

$$\phi(t, \vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \frac{1}{\sqrt{2E_p}}(a_{\vec{p}}(t)e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}(t)e^{-ip\cdot x}).$$

If we're in a spatial box, then $p_i L_i = 2\pi n_i$. Compute the energy to find

$$H = \sum_{\vec{p}>0}(\frac{1}{2E_p}\dot{a}^\dagger_p\dot{a}_p(t) + \tfrac{1}{2}E_p a^\dagger_p a_p) = \sum_{\vec{p}} E_p a^\dagger_{\vec{p}} a_{\vec{p}}.$$

where the EOM were used in the last step: $a_{\vec{p}}(t) = a_{\vec{p}}e^{-iE_pt} + a^\dagger_{-\vec{p}}e^{iE_pt}$. Also,

$$\vec{P} = \sum_{\vec{p}} \vec{p} a^\dagger_{\vec{p}} a_{\vec{p}}.$$

As expected, $H$ and $\vec{P}$ are independent of $t$. We quantize this as a (complex) SHO for each value of $\vec{p}$:

$$[a_p, a_k^\dagger] = \delta_{p,k}, \qquad [a_p, a_k] = [a_p^\dagger, a_k^\dagger] = 0.$$

and interpret the above $H$ and $\vec{P}$ has saying that $a_{\pm\vec{p}}^\dagger$ is a creation operator, creating a state with energy $E_p = \sqrt{\vec{p}^2 + m^2}$ and spatial momentum $\vec{p}$ from the vacuum $|\Omega\rangle$. Note that we dropped the $2 \cdot \frac{1}{2} E_p$ groundstate energy contribution, for no good reason. This is a contribution to the vacuum energy of empty space, and it is divergent upon summing over all $p$. This zero point energy is important (only) for gravity, and a contribution to the cosmological constant. Since this is an unresolved problem, we won't discuss it further.

• Now consider the Maxwell field $A^\mu$ and quantize $\rightarrow$ photons. In the vacuum, setting $j^\mu = 0$, we have $\partial_\mu F^{\nu\mu} = 0$, which implies $\partial^2 A^\mu - \partial^\mu(\partial \cdot A) = 0$. Massless. Fourier transform to $A^\mu(p)$, with $A^\mu(-p) = A^\mu(p)^*$, and get $(p^2\eta^{\mu\nu} - p^\mu p^\nu)A_\nu(p) = 0$. Gauge invariance: $\delta A_\mu(p) = ip_\mu \epsilon(p)$. In light cone gauge, since $p^+ \neq 0$, can use gauge invariance to choose $\epsilon$ such that $A^+(p) = 0$. Then taking $\mu = +$ in the EOM, and $p^+ \neq 0$, get $\partial \cdot A = 0$ which gives $A^- = (p^I A^I)/p^+$, i.e. $A^-$ is not an independent d.o.f., but rather constrained, and the Maxewell EOM gives $p^2 A^\mu(p) = 0$. For $p^2 \neq 0$, require $A^\mu(p) = 0$, and for $p^2 = 0$ get that there are $D - 2$ physical transverse d.o.f., the $A^I(p)$. The one-photon states are

$$\sum_{I=2}^{D-1} \xi_I a_{p^+, p_T}^{I\dagger} |\omega\rangle.$$