5/14/19 Lecture outline

$\star$ Reading: Zwiebach chapter 9.

• As some review for the next topic, recall the light cone coordinates: $a^\pm = (a^0 \pm a^1)/\sqrt{2}$, so $a\cdot b = -a^-a^+ - a^+a^- + \sum_I a^I b^I$, where $I = 2, \ldots$ runs over the transverse space directions. Ugly, but can help quantize. Recall the light cone metric gives $a^\pm = -a_\mp$. Now recall a plane wave solution of QM $\psi = e^{ip_\mu x^\mu/\hbar}$, which has $E \to i\hbar\partial_t$ and $\vec p \to -i\hbar\nabla$. In light cone coordinates, we write it as $\psi = \exp(i(-p^- x^+ - p^+ x^- + \sum_I p_I x^I)/\hbar)$. We think of $x^+$ as time and then see that $p^-$ is the energy.

In non-relativisitic QM, get Schrodinger equation by writing $H = \vec p^2/2m + V(x)$ and replacing $H \to i\hbar\frac{\partial}{\partial t}$ and $\vec p \to -i\hbar\nabla$. In relativistic case, considering free particle for simplicity, have $H = \sqrt{(c\vec p)^2 + (mc^2)^2}$ and $\vec p \to -i\hbar\nabla$ would require understanding how to take the square-root of an operator. This is what led Dirac to the Dirac equation for relativistic electrons, and the start of quantum field theory. Very interesting, but not the topic of this class. We will avoid going there by the trick of the light cone.

Write $-p\cdot p = m^2$ (setting $c = 1$) as $2p^+p^- = \sum_I p^I p^I + m^2$. In the light cone, we think of $x^+$ as time. Then $p^- = H_{lc} = (\sum_I p^I p^I + m^2)/2p^+$. No need for square-root. Looks similar to non-relativistic case.

• Use $\hbar = c = 1$ units. Recall $[\alpha'] = 1/[T_0] = L^2$. Write

$$\mathcal{L}_{NG} = -\frac{1}{2\pi\alpha'}\sqrt{(\dot X\cdot X')^2 - (\dot X)^2(X')^2},$$

and we have

$$\mathcal{P}^\tau_\mu = \frac{\partial\mathcal{L}}{\partial\dot X^\mu} = -\frac{1}{2\pi\alpha'}\frac{(\dot X\cdot X')X'_\mu - (X')^2\dot X_\mu}{\sqrt{(\dot X\cdot X')^2 - (\dot X)^2(X')^2}},$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial\mathcal{L}}{\partial X^{\mu\prime}} = -\frac{1}{2\pi\alpha'}\frac{(\dot X\cdot X')\dot X_\mu - (\dot X)^2 X'_\mu}{\sqrt{(\dot X\cdot X')^2 - (\dot X)^2(X')^2}}.$$

which we simplified by picking static gauge.

• Generalize static gauge (to eventually get to light cone gauge). Consider e.g. gauge $n_\mu X^\mu = \lambda\tau$ for time-like $n_\mu$, so our previous static gauge is $n_\mu = (1, 0, \ldots, 0)$. Light cone static gauge takes instead $n_\mu = \frac{1}{\sqrt{2}}(1, 1, 0\ldots)$. Vary, $n_\mu dX^\mu = \lambda d\tau$, so $n_\mu$ is orthogonal to the string tangent at constant $\tau$. We want $dX^\mu$ along the string to be spacelike (or null at isolated points, e.g. the Neumann open string endpoints).

Before, we took $(\tau, \sigma)$ to have units of time and length. Now it is more convenient to take them to be dimensionless, and use $\lambda$ to account for the units. As we will see, for open strings it is natural to take $\sigma \in [0, \pi]$ and $\lambda = 2\alpha'(n \cdot p)$. Note that the units work, and that the upshot is invariant under rescaling $n_\mu$ and that $n \cdot p$ is a constant of the motion. We will see that for closed strings it is nicer to instead take $\sigma \in [0, 2\pi]$ and $\lambda = \alpha'(n \cdot p)$. We're free to choose these normalizations conveniently because we can rescale $\tau$ to make it work, and that is a subgroup of the full reparameterization symmetry.

Before, we chose our $\sigma$ parameterization such that $n_\mu \mathcal{P}^{\tau\mu}$ is a constant. We will likewise, for general $n^\mu$, choose $\sigma$ such that $n \cdot \mathcal{P}^\tau$ is a constant of the motion of the string worldsheet. This is not a reprarameterization invariant statement - that is the point: we are using it to fix a gauge. Using the EOM, this implies that $n \cdot \mathcal{P}^\sigma$ is independent of $\sigma$ and then can argue that $n \cdot \mathcal{P}^\sigma = 0$.

More generally, it is convenient to write the gauge fixing conditions as

$$n \cdot \mathcal{P}^\sigma = 0, \qquad n \cdot X = \beta\alpha'(n \cdot p)\tau, \qquad n \cdot p = \frac{2\pi}{\beta} n \cdot \mathcal{P}^\tau,$$

where $\beta = 2$ for open strings and $\beta = 1$ for closed strings. These lead to

$$\dot{X} \cdot X' = 0 \qquad \dot{X}^2 + c^2 X'^2 = 0. \tag{1}$$

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu \qquad \mathcal{P}^{\sigma\mu} = -\frac{c^2}{2\pi\alpha'} X^{\mu'}, \tag{2}$$

$$(\partial_\tau^2 - c^2 \partial_\sigma^2) X^\mu = 0. \tag{3}$$

• We will later focus on light cone gauge: $n_\mu = (1/\sqrt{2}, 1/\sqrt{2}, 0, \ldots)$. Introducing $n^\mu$ obscures the relativistic invariance in spacetime. Why would we want to do that? Well we wouldn't, except that it happens to have some other benefits once we quantize the theory. It gives a way to determine the spectrum without having to introduce unphysical states. There is a covariant approach, but it requires introducing unphysical states ("ghosts") and then ensuring that they are projected out of the physical spectrum – doing this requires sophisticated theory which is only taught at the advanced graduate student level, so we'll stick with the simpler (and in the end physically equivalent) light-cone gauge description.

• The general solution of the linear equations (3) is a superposition of Fourier modes

$$X^\mu(\tau, \sigma) = x_0^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0}^{\infty} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma,$$

where $\alpha^\mu_{-n} \equiv \alpha^{\mu *}_n$ (to make $X^\mu$ real) and it's also convenient to define $\alpha^\mu_0 \equiv \sqrt{2\alpha'}p^\mu$. Then

$$\dot{X}^\mu \pm X^{\mu'} = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \alpha^\mu_n e^{-in(\tau\pm\sigma)}.$$

• In light cone gauge take $n_\mu = (1/\sqrt{2}, 1/\sqrt{2}, 0, \ldots)$. Then $n \cdot X = X^+$ and $n \cdot p = p^+$, so our constraint gives $X^+ = \beta\alpha' p^+ \tau$ and $p^+ = 2\pi \mathcal{P}^{\tau +}/\beta$ (again, $\beta = 2$ for open strings and $\beta = 1$ for closed strings. Also note, $X'^+ = 0$ and $\dot{X}^+ = \beta\alpha' p^+$); of course, $p^+$ is a constant of the motion. Since the constraints give $(\dot{X} \pm X')^2 = -2(\dot{X}^+ \pm X'^+)(\dot{X}^- \pm X'^-) + (\dot{X}^I \pm X'^I)^2 = 0$, we can write this as $\partial_\tau X^- \pm \partial_\sigma X^- = \frac{1}{\beta\alpha'}\frac{1}{2p^+}(\dot{X}^I \pm X^{I'})^2$, where $I$ are the transverse directions. This leads to

$$\sqrt{2\alpha'}\alpha^-_n \equiv \frac{1}{p^+} L^\perp_n, \qquad L^\perp_n = \tfrac{1}{2} \sum_{m=-\infty}^{\infty} \alpha^I_{n-m} \alpha^I_m.$$

Note that the worldsheet coordinates dropped out. This means that there is no dynamics in $X^-$, other than the zero mode.