5/9/19 Lecture outline

⋆ Reading: Zwiebach chapters 8, 9.

• Continue where we left off last time: symmetry and conservation laws for the case of Lorentz transformations. As we discussed, continuous symmetries $\delta\phi^a$ of a field theory with $S = \int d\xi^0 \dots d\xi^{p-1}\mathcal{L}(\phi^a, \partial_\alpha\phi^a)$ leads to conserved currents $j^\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\delta\phi^a$, satisfying $\partial_\alpha j^\alpha = 0$. We apply this to the case of the worldsheet, where symmetries are δX^μ and $\alpha = 0, 1 \to \tau, \sigma$. The corresponding worldsheet conserved charge is $\int d\sigma j^\tau$. More generally, we have conserved charge $\int_\Gamma (j^\tau d\sigma - j^\sigma d\tau)$, where $(d\sigma, -d\tau)$ is the outward normal to a curve with tangent $(d\tau, d\sigma)$. If the curve is closed, we can use Stokes theorem to get $\oint_\Gamma (j^\tau d\sigma - j^\sigma d\tau) = \int_R (\partial_\tau j^\tau + \partial_\sigma j^\sigma) d\tau d\sigma = 0$ by current conservation, showing again that the charge is conserved and independent of deformations of the curve Γ.

For $\delta X^\mu = \epsilon^\mu$ translations, the charge is $p_\mu = \int d\sigma \mathcal{P}^\tau_\mu$. Now continue with Lorentz symmetry, which comes from the worldsheet symmetry $\delta X^\mu = \epsilon^{\mu\nu}X_\nu$, which is a symmetry if $\epsilon^{\mu\nu} = \epsilon^{[\mu\nu]}$, e.g. $\delta(\eta_{\mu\nu}X^\mu X^\nu) = 0$. Discuss cases of spatial rotations and boosts, explain why both indeed involve antisymmetric $\epsilon^{\mu\nu}$. Lorentz symmetry is of course a symmetry of the string Lagrangian $\mathcal{L}_{NG} = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}$ involve $\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$, since all Lorentz vector indices are contracted via Lorentz scalar dot products.

The associated conserved currents are $\mathcal{M}^\alpha_{\mu\nu} = X_\mu \mathcal{P}^\alpha_\nu - (\mu \leftrightarrow \nu)$. The corresponding charges $M_{\mu\nu} = \int (\mathcal{M}^\tau_{\mu\nu} d\sigma$ are the angular momentum. We can also consider more generally conserved charges $M_{\mu\nu}[\Gamma] = \int_\Gamma (\mathcal{M}^\tau_{\mu\nu} d\sigma - \mathcal{M}^\sigma_{\mu\nu} d\tau)$. Note that the charges associated with boosts are $M^{0i} = ctp^i - \int d\sigma X^i \mathcal{P}^{\tau 0}$, which can be interpreted as $X^i_{cm}(t) = \frac{-cM^{0i}}{E} + t\frac{c^2 p^i}{E}$.

• Recall $J = \hbar\alpha' E^2$, with $[\alpha'] = -2$, which is the Regge trajectory observation of the early '70s.. Consider now a string rotating in 12 plane, with the EOM solved by (as discussed last week): $\vec{X} = \frac{\sigma_1}{\pi}\cos(\pi\sigma/\sigma_1)(\cos(\pi ct/\sigma_1), \sin(\pi ct/\sigma_1))$. So $\vec{\mathcal{P}}^\tau = \frac{T_0}{c^2}\dot{\vec{X}} = \frac{T_0}{c}\cos(\pi\sigma/\sigma_1)(-\sin(\pi ct/\sigma_1), \cos(\pi ct/\sigma_1))$. Find that the rotational angular momentum has $M_{12} = \int_0^{\sigma_1} d\sigma(X_1\mathcal{P}^\tau_2 - X_2\mathcal{P}^\tau_1)$, which using above $\vec{X}(t,\sigma)$ and $\vec{\mathcal{P}}^\tau = \frac{T_0}{c^2}\partial_t\vec{X}$, leads to $M_{12} = \sigma_1^2 T_0/2\pi c$, which is a constant as expected. Since $\sigma_1 = E/T_0$ and $M_{12} = J$, this gives $J = \alpha'\hbar E^2$, with $T_0 \equiv 1/2\pi\alpha'\hbar c$. The string length is $\ell_s = \hbar c\sqrt{\alpha'}$.

• Aside, for later: the string worldsheet analog of $S_{particle} \supset \int qA_\mu dx^\mu$ is $S_{string} \supset -\int_\Sigma B_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X^\nu d\sigma d\tau$.

• Outline of plan, to quantize the relativistic string. Description of challenge of quantizing $\mathcal{L}_{NG}$ because of the square-root. Recall how the trying to quantize with the square-root in $E = \sqrt{(cp)^2 + (mc^2)^2}$ led Dirac to the Dirac equation. Summarize two approaches: the Polyakov description or using light-cone gauge quantization. This class will follow the latter approach, as in the textbook.