5/2/19 Lecture outline

⋆ Reading: Zwiebach chapters 7, 8.

• Continue where we left off. We chose a gauge such that the NG string EOM simplifies

$$(\dot{X} \pm X')^2 = 0 \qquad \to \qquad (\partial_\tau^2 - c^2\partial_\sigma^2)X^\mu = 0 \tag{1}$$

with $d\sigma = \frac{ds}{\sqrt{1-v_\perp^2/c^2}} = \frac{dE}{T_0}$.

We were studying the solution of the EOM for open string with free (N) BCs at each end. Write the solution of the EOM as $\vec{X}(t,\sigma) = \frac{1}{2}(\vec{F}(ct+\sigma) + G(ct-\sigma))$. The BC at $\sigma = 0$ gives $F'(ct) = G'(ct)$, which implies $G = F + const$, and the constant can be absorbed into F so $\vec{X}(t,\sigma) = \frac{1}{2}(\vec{F}(ct+\sigma) + \vec{F}(ct-\sigma))$ where the open string has $\sigma \in [0, \sigma_1]$ and the constraints in (1) imply that $|\frac{d\vec{F}(u)}{du}|^2 = 1$, and $\vec{X}'|_{ends} = 0$ implies $\vec{F}(u + 2\sigma_1) = \vec{F}(u) + 2\sigma_1\vec{v}_0/c$. Note $\vec{F}(u)$ is the position of the $\sigma = 0$ end at time u/c. Then show that $\vec{v}_0$ is the average velocity of any point σ on the string over time interval $2\sigma_1/c$. Observing motion of $\sigma = 0$ end over that Δt, together with E, gives motion of string for all t. Example from book: $\vec{X}(t, \sigma = 0) = \frac{\ell}{2}(\cos\omega t, \sin\omega t)$. Find $\vec{F}(u) = \frac{\sigma_1}{\pi}(\cos\pi u/\sigma_1, \sin\pi u/\sigma_1)$, with $\vec{v}_0 = 0$. $|\frac{d\vec{F}}{du}|^2 = 1$ gives $\ell = 2c/\omega = 2E/\pi T_0$. Finally, $\vec{X}(t,\sigma) = \frac{\sigma_1}{\pi}\cos(\pi\sigma/\sigma_1)(\cos(\pi ct/\sigma_1), \sin(\pi ct/\sigma_1))$. Note that the ends indeed move at the speed of light.

• Closed string motion: again, solve the string worldsheet wave equation by $\vec{X} = \frac{1}{2}(\vec{F}(u) + \vec{G}(v))$ where $u = ct + \sigma$ and $v = ct - \sigma$. The parameterization constraints give $|\vec{F}'(u)|^2 = |\vec{G}'(v)|^2 = 1$ and $\sigma \sim \sigma + \sigma_1$ periodicity, with $\sigma_1 = E/T_0$.

• Next topic: symmetries and conservation laws, on the string worldsheet and in spacetime. Recall charge conservation $\partial_\mu j^\mu = 0$, which is required by gauge invariance of $\mathcal{L} \supset A_\mu j^\mu$, i.e. $\delta\mathcal{L} = 0$ under $\delta A_\mu = \partial_\mu f$. Show that it implies conservation of $Q = \int d^3x j^0$.

• Recall Noether's theorem for $L(q, \dot{q})$: continuous symmetry δq_i implies that $p_i \delta q_i$ is conserved.

Likewise, for $S = \int d\xi^0 \dots d\xi^k \mathcal{L}(\phi^a, \partial_\alpha\phi^a)$, a symmetry $\delta\phi^a$ implies a conserved current $j^\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\delta\phi^a$: show it satisfies $\partial_\alpha j^\alpha = 0$, so $Q = \int d\xi^1 \dots d\xi^k$ has $\frac{d}{d\xi^0}Q = 0$.

For a string $S = \int d\xi^0 d\xi^1 \mathcal{L}(\partial_\alpha X^\mu)$ (has translation invariance, $\delta X^\mu = \epsilon^\mu$ so there is a conserved current $\epsilon^\mu j^\alpha_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha X^\mu)}\delta X^\mu$. So get conservation of $j^a_\mu = \mathcal{P}^a_\mu$ (where $a = \sigma, \tau$) is the conserved Noether current for spacetime translation invariance, $\delta X^\mu = \epsilon^\mu$. The string equations of motion are equivalent to the worldsheet conservation of this current: $\partial_a j^a_\mu = 0$. The spacetime momentum of the string is the corresponding conserved charge:

$p_\mu = \int d\sigma \mathcal{P}^\tau_\mu$. So $\frac{dp_\mu}{d\tau} = -\int_0^{\sigma_1} \partial_\sigma \mathcal{P}^\sigma_\mu = -\mathcal{P}^\sigma_\mu|_0^{\sigma^1}$. It is conserved for the closed string, or open Neumann BCs. Not conserved for Dirichlet BCs. The Dirichlet case means that the string ends on a D-brane, and momentum can go through the string into the D-brane (their total momentum is conserved). Same for wave on a string with the ends tied down, e.g. a traveling wave is reflected, which flips $p \to -p$, but the difference in momentum is transferred to the post at the end and total momentum of the system is conserved.

• Note that p_μ is a conserved *worldsheet* charge. It becomes a conserved spacetime charge in static gauge, $\tau = t$. We can write more generally the conserved flux of worldsheet current as $(\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu) \cdot (d\sigma, -d\tau)$, where $(d\tau, d\sigma)$ is the tangent to the curve Γ that we're integrating over and $(d\sigma, -d\tau)$ gives the outward normal. So $p_\mu(\Gamma) = \int_\Gamma (\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau)$. The difference between some Γ and Γ' with the same endpoints (i.e. $\partial(\Gamma - \Gamma') = 0$) is $\oint_{\Gamma-\Gamma'=\partial R} (\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau) = \int_R d\tau d\sigma (\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu) = 0$.

• Using $\mathcal{P}^{\alpha\mu}$ in static gauge, we get for the conserved charges

$$p^0 = \frac{E}{c} = \int \frac{T_0 ds}{\sqrt{1 - v_\perp^2/c^2}}, \qquad \vec{p} = \int \frac{T_0 ds}{c^2} \frac{v_\perp}{\sqrt{1 - v_\perp^2/c^2}}.$$

• Lorentz symmetry comes from the worldsheet symmetry $\delta X^\mu = \epsilon^{\mu\nu} X_\nu$, which is a symmetry if $\epsilon^{\mu\nu} = \epsilon^{[\mu\nu]}$, e.g. $\delta(\eta_{\mu\nu} X^\mu X^\nu) = 0$. The terms in the string Lagrangian $\mathcal{L}_{NG} = -\frac{T_0}{c}\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}$ involve $\eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu$, which again is invariant under $\delta X^\mu = \epsilon^{\mu\nu} X_\nu$.

The associated conserved currents are $\mathcal{M}^\alpha_{\mu\nu} = X_\mu \mathcal{P}^\alpha_\nu - (\mu \leftrightarrow \nu)$. The corresponding charges $M_{\mu\nu} = \int (\mathcal{M}^\tau_{\mu\nu} d\sigma - \mathcal{M}^\sigma_{\mu\nu} d\tau)$ are the angular momenta. Note that $M^{0i} = ctp^i - \int d\sigma X^i \mathcal{P}^{\tau 0}$, which can be interpreted as $X^i_{cm}(t) = \frac{-cM^{0i}}{E} + t\frac{c^2 p^i}{E}$.

• Recall $J = \hbar\alpha' E^2$, with $[\alpha'] = -2$, which is the Regge trajectory observation of the early '70s.. Consider now a string rotating in 12 plane, with the EOM solved by $\vec{X} = \frac{\sigma_1}{\pi}\cos(\pi\sigma/\sigma_1)(\cos(\pi ct/\sigma_1), \sin(\pi ct/\sigma_1))$. So $\vec{\mathcal{P}}^\tau = \frac{T_0}{c^2}\dot{\vec{X}} = \frac{T_0}{c}\cos(\pi\sigma/\sigma_1)(-\sin(\pi ct/\sigma_1), \cos(\pi ct/\sigma_1))$. Find that the rotational angular momentum has $M_{12} = \int_0^{\sigma_1} d\sigma (X_1 \mathcal{P}^\tau_2 - X_2 \mathcal{P}^\tau_1)$, which using above $\vec{X}(t,\sigma)$ and $\vec{\mathcal{P}}^\tau = \frac{T_0}{c^2}\partial_t \vec{X}$, leads to $M_{12} = \sigma_1^2 T_0/2\pi c$, which is a constant as expected. Since $\sigma_1 = E/T_0$ and $M_{12} = J$, this gives $J = \alpha'\hbar E^2$ $\ell_s = \hbar c\sqrt{\alpha'}$, with $T_0 \equiv 1/2\pi\alpha'\hbar c$.

• Aside, for later: the string worldsheet analog of $S_{particle} \supset \int qA_\mu dx^\mu$ is $S_{string} \supset -\int_\Sigma B_{\mu\nu} \partial_\tau X^\mu \partial_\sigma X^\nu d\sigma d\tau$.