5/1/18 Lecture outline

⋆ Reading: Zwiebach chapters 6,7

• Recall from last time:

$$\mathcal{L}_{NG} = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2},$$

and we have

$$\mathcal{P}^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}},$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}.$$

The condition $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0.$$

Exploit $(\tau,\sigma)\to(\tau',\sigma')$ reparameterization invariance to pick useful "gauges", to simplify the above equations. We will eventually choose such that we can impose constraints

$$\dot{X}\cdot X' = 0 \qquad \dot{X}^2 + X'^2 = 0. \tag{1}$$

In this case, we have

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu \qquad \mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}X^{\mu\prime}, \tag{2}$$

and then the EOM is simply a wave equation:

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0. \tag{3}$$

• We will motivate the above choice by discussing in more detail the interpretation of $X^\mu(\tau,\sigma)$. Consider the tangent vectors $\partial_\tau X^\mu$ and $\partial_\sigma X^\mu$; aside from isolated points, we can and will choose τ and σ such that they are timeline and space-like, respectively. Take $v^\mu(\lambda) = \partial_\tau X^\mu + \lambda\partial_\sigma X^\mu$, so $v^2 = (\dot{X})^2 + 2\lambda\dot{X}\cdot X' + \lambda^2(X')^2$ which can be either positive or negative, so there must be two real λ solutions to the condition $v^2 = 0$; the condition that this is true is that the descriminant of the quadratic equation must be positive, and that is precisely what is inside the $\sqrt{\cdot}$ in $\mathcal{L}_{NG}$.

Since $\dot{X}^\mu$ is timelike, we can choose static gauge, where $\tau = t$. Verify sign inside $\sqrt{\cdot}$ in this case: $X^{\mu\prime} = (0, \vec{X}')$, $\dot{X}^\mu = (c, \dot{\vec{X}})$, take e.g. $\dot{\vec{X}} = 0$ to get $\sqrt{\cdot} = c|\vec{X}'|$.

• Consider example of $X^\mu(\sigma,\tau) = (c\tau, f(\sigma), 0, \ldots 0)$. So $\dot{X}^\mu = (c, \vec{0})$ and $X'^\mu = (0, f'(\sigma), 0, \ldots, 0)$. Verify that the EOM are satisfied. Compute the action and note that $V = T_0 a$ where a is the length of the string.

• In static gauge, let $ds \equiv |d\vec{X}|_{t=const} = |\partial_\sigma \vec{X}||d\sigma|$ be the length element of the string for σ varying over $d\sigma$ at fixed $t = \tau$. Note that $\partial_s \vec{X}$ is a unit vector, which is spacelike since it is along $d\sigma$, i.e. along the string. The transverse velocity to the string is the component of $\partial_t \vec{X}$ that is perpendicular to this unit vector: $\vec{v}_\perp = \partial_t \vec{X} - (\partial_t \vec{X} \cdot \partial_s \vec{X})\partial_s \vec{X}$.

• Note that $(\dot{X}\cdot X')^2 - \dot{X}^2 (X')^2 = (\frac{ds}{d\sigma})^2 (c^2 - v_\perp^2)$, so S_{NG} has $L_{NG} = -T_0 \int ds \sqrt{1 - v_\perp^2/c^2}$. This fits with $L_{rel,pp} = -mc\sqrt{1 - v^2/c^2}$. Note that

$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c^2} \frac{(\partial_s \vec{X} \cdot \partial_t \vec{X})\dot{X}^\mu + (c^2 - (\partial_t \vec{X})^2)\partial_s X^\mu}{\sqrt{1 - v_\perp^2/c^2}},$$

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2} \frac{ds}{d\sigma} \frac{\dot{X}^\mu - (\partial_s \vec{X} \cdot \partial_t \vec{X})\partial_s X^\mu}{\sqrt{1 - v_\perp^2/c^2}}.$$

• Free, Neuman BCs, P^σ_μ for the $\mu = 0$ component implies that endpoints move transversely, $\partial_s \vec{X} \cdot \partial_t \vec{X} = 0$, so $\vec{v}_\perp = \vec{v}$. The condition $\vec{P}^\sigma = 0$ at the endpoints implies that the speed of light, $v = c$, for the free (Neuman) BCs.

• Step 2 (Z, chapter 7): we can choose σ such that $\partial_\sigma \vec{X} \cdot \partial_t \vec{X} = 0$ along entire string (we saw it above for Neumann endpoints). The interpretation is that we take the timelike and spacelike vectors $\dot{X}^\mu$ and X'^μ to be orthogonal. This gives $\vec{v}_\perp = \vec{v} \equiv \dot{\vec{X}}$ along the entire string. Then $\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\frac{ds}{d\sigma}\gamma \partial_t X^\mu$ and $\mathcal{P}^{\sigma\mu} = -T_0 \gamma^{-1} \partial_s X^\mu$, with $\gamma \equiv 1/\sqrt{1 - v_\perp^2/c^2}$.

Now consider the $\mu = 0$ component of the EOM: $\partial_t \mathcal{P}^{\tau\mu} = -\partial_\sigma \mathcal{P}^{\sigma\mu}$, which for $\mu = 0$ gives that $(T_0/c)\frac{ds}{d\sigma}\gamma$ is a constant of the motion. Indeed this is proportional to the energy of an element of string. In a HW you will show that the string Hamiltonian is indeed $H = \int T_0 ds/\sqrt{1 - v_\perp^2/c^2}$.

Now the space components of the EOM can be written as $\mu_{eff}\partial_t \vec{v}_\perp = \partial_s (T_{eff} \partial_s \vec{X})$, with $T_{eff} = T_0/\gamma$ and $\mu_{eff} = T_0 \gamma / c^2$.

• Now note that since $\frac{ds}{d\sigma}\gamma$ is a constant, we can set it equal to 1. This can be written as the constraint: $(\partial_\sigma \vec{X})^2 + (\partial_{X_0} \vec{X})^2 = 1$.

• Summary: choose σ parameterization such that

$$\partial_\sigma \vec{X} \cdot \partial_\tau \vec{X} = 0 \qquad \text{and} \qquad d\sigma = \frac{ds}{\sqrt{1 - v_\perp^2/c^2}} = \frac{dE}{T_0}.$$

(Using $H = \int T_0 ds/\sqrt{1 - v_\perp^2/c^2}$ and $\partial_t(ds/\sqrt{1 - v_\perp^2/c^2}) = 0$.) The last equation above is equivalent to $(\partial_\sigma \vec{X})^2 + c^{-2}(\partial_t \vec{X})^2 = 1$. With this worldsheet gauge choice,

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\partial_t X^\mu = \frac{T^0}{c^2}(c, \vec{v}_\perp), \qquad \mathcal{P}^{\sigma,\mu} = -T_0 \partial_\sigma X^\mu = (0, -T_0 \partial_\sigma \vec{X}).$$

We can write this as

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu \qquad \mathcal{P}^{\sigma\mu} = -\frac{c^2}{2\pi\alpha'}X^{\mu'}, \tag{4}$$

and then the EOM is simply a linear wave equation, and we also need to impose the constraints:

$$(\partial_\tau^2 - c^2\partial_\sigma^2)X^\mu = 0, \qquad (\dot{X} \pm X')^2 = 0. \tag{5}$$

• Solution of the EOM for open string with free (N) BCs at each end. Write the solution of the EOM as $\vec{X}(t,\sigma) = \frac{1}{2}(\vec{F}(ct+\sigma) + G(ct-\sigma))$. The BC at $\sigma = 0$ gives $F'(ct) = G'(ct)$, which implies $G = F + const$, and the constant can be absorbed into F so $\vec{X}(t,\sigma) = \frac{1}{2}(\vec{F}(ct+\sigma) + \vec{F}(ct-\sigma))$ where the open string has $\sigma \in [0, \sigma_1]$ and (1) implies that $|\frac{d\vec{F}(u)}{du}|^2 = 1$, and $\vec{X}'|_{ends} = 0$ implies $\vec{F}(u + 2\sigma_1) = \vec{F}(u) + 2\sigma_1\vec{v}_0/c$. Note $\vec{F}(u)$ is the position of the $\sigma = 0$ end at time u/c. Then show that $\vec{v}_0$ is the average velocity of any point σ on the string over time interval $2\sigma_1/c$. Observing motion of $\sigma = 0$ end over that Δt, together with E, gives motion of string for all t. Example from book: $\vec{X}(t, \sigma = 0) = \frac{\ell}{2}(\cos\omega t, \sin\omega t)$. Find $\vec{F}(u) = \frac{\sigma_1}{\pi}(\cos\pi u/\sigma_1, \sin\pi u/\sigma_1)$, with $\vec{v}_0 = 0$. $|\frac{d\vec{F}}{du}|^2 = 1$ gives $\ell = 2c/\omega = 2E/\pi T_0$. Finally, $\vec{X}(t,\sigma) = \frac{\sigma_1}{\pi}\cos(\pi\sigma/\sigma_1)(\cos(\pi ct/\sigma_1), \sin(\pi ct/\sigma_1))$.