4/24/18 Lecture outline

⋆ Reading: Zwiebach chapters 4,5,6.

• Continue from last time: mass m is $S=-mc\int ds=-mc\int\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$, proportional to the worldline length and reparameterization invariant under $\tau\to\tau'(\tau)$. Likewise, for a string world-sheet, we need two parameters, ξ^a, $a=1,2$. The string trajectory is $x:\Sigma\to M$, where Σ is the 2d world-sheet, with local coordinates ξ^a, and M is the target space, with local coordinates x^μ. The worldsheet area element is $A=\int d^2\xi\sqrt{|h|}$, where h_{ab} is the worldsheet metric, and $|h|$ is its determinant. Suppose that the target space has metric $g_{\mu\nu}$, with space-time length e.g. $ds^2=g_{\mu\nu}dx^\mu dx^\nu$. By writing $dx^\mu=\partial_a x^\mu d\xi^a$, we get

$$ds^2=g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b}d\xi^a d\xi^b, \qquad \text{so} \qquad h_{ab}=g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b},$$

where this h_{ab} is called the induced metric. So the worldsheet area functional is

$$A=\int d^2\xi\sqrt{\det_{ab}(g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b})}.$$

For strings in Minkowski spacetime, we write it instead as $X^\mu(\tau,\sigma)$. There is also a needed minus sign, as the area element is $\sqrt{|g|}$, actually involves the absolute value of the determinant, and the determinant is negative (just like $\det\eta=-1$). So

$$A=\int d\tau d\sigma\sqrt{(\frac{\partial X}{\partial\tau}\cdot\frac{\partial X}{\partial\sigma})^2-(\frac{\partial X}{\partial\tau})^2(\frac{\partial X}{\partial\sigma})^2},$$

where the spacetime indices are contracted with the metric $g_{\mu\nu}$. To get an action with $[S]=ML^2/T$, we have

$$S_{Nambu-Goto}=-\frac{T_0}{c}\int_{\tau_i}^{\tau_f}d\tau\int d\sigma\sqrt{(\dot{X}\cdot X')^2-(\dot{X})^2(X')^2},$$

where we define $\dot{X}^\mu\equiv\frac{dx^\mu}{d\tau}$ and $X^{\mu\prime}\equiv\frac{\partial X^\mu}{\partial\sigma}$ annd T_0 is the string tension, with $[T_0]=[F]=[ML/T^2]$.

The action is reparameterization invariant: can take $(\tau,\sigma)\to(\tau'(\tau,\sigma),\sigma'(\tau,\sigma))$ and get $S\to S$. Enormous symmetry/redundancy in choice of (τ,σ); can "fix the gauge" to some convenient choice, and the physics is completely independent of the choice. This is crucial, since the worldsheet coordinates have no physical significance.

• We can write $S_{NG}=\int d^2\xi\mathcal{L}_{NG}$ with Lagrangian density

$$\mathcal{L}_{NG}=-\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2-(\dot{X})^2(X')^2},$$

and we have

$$\mathcal{P}^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}},$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}.$$

The condition $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0.$$

For the open string, $\delta S = 0$ also requires $\int d\tau [\delta X^\mu P^\sigma_\mu]_0^{\sigma_0} = 0$, which requires for each μ index either of the Dirichlet or Neumann BCs, at each end:

$$\text{Dirichlet} \qquad \frac{\partial X^\mu}{\partial \tau}(\tau, \sigma_*) = 0 \qquad \to \qquad \delta X^\mu(\tau,\sigma_*) = 0,$$

$$\text{Neumann} \qquad \mathcal{P}^\sigma_\mu(\tau,\sigma_*) = 0.$$

• Exploit $(\tau,\sigma) \to (\tau',\sigma')$ reparameterization invariance to pick useful "gauges", to simplify the above equations. We will discuss choices such that we can impose constraints

$$\dot{X}\cdot X' = 0 \qquad \dot{X}^2 + X'^2 = 0. \tag{1}$$

In this case, we have

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu \qquad \mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}X^{\mu\prime}, \tag{2}$$

and then the EOM is simply a wave equation:

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0. \tag{3}$$

Now let's explain these things in more detail.

• Static gauge: pick $\tau = t$. Verify sign inside $\sqrt{\cdot}$ in this case: $X^{\mu'} = (0, \vec{X}')$, $\dot{X}^\mu = (c, \dot{\vec{X}})$, take e.g. $\dot{\vec{X}} = 0$ to get $\sqrt{\cdot} = c|\vec{X}'|$.

• In static gauge, there is no KE, so $L = -V$, and verify that string stretched length a, e.g. $X^1 = f(\sigma)$, has $V = T_0 a$: $\dot{X}^2 \to -c^2$, $(X')^2 = (f')^2$, $\dot{X}\cdot X' = 0$, gives $V = T_0 a$. So $\mu_0 = T_0/c^2$.

• In static gauge, express S in terms of $\vec{v}_\perp = \partial_t \vec{X} - (\partial_t \vec{X}\cdot\partial_s\vec{X})\partial_s\vec{X}$ (with $ds \equiv |d\vec{X}|_{t=const} = |\partial_\sigma \vec{X}||d\sigma|$), show $(\dot{X}\cdot X')^2 - \dot{X}^2(X')^2 = (\frac{ds}{d\sigma})^2(c^2 - v_\perp^2)$, to get $L = -T_0\int ds\sqrt{1 - v_\perp^2/c^2}$. Also get

$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c^2}\frac{(\partial_s\vec{X}\cdot\partial_t\vec{X})\dot{X}^\mu + (c^2 - (\partial_t\vec{X})^2)\partial_s X^\mu}{\sqrt{1 - v_\perp^2/c^2}},$$

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\frac{ds}{d\sigma}\frac{\dot{X}^\mu - (\partial_s\vec{X}\cdot\partial_t\vec{X})\partial_s X^\mu}{\sqrt{1 - v_\perp^2/c^2}}.$$