

4/24/18 Lecture outline

- ★ Reading: Zwiebach chapters 4,5,6.
- Nonrelativistic strings.  $\frac{\partial^2 y}{\partial x^2} - \frac{1}{v_0^2} \frac{\partial^2 y}{\partial t^2} = 0$ , with  $v_0 = \sqrt{T_0/\mu_0}$ .
- We can get the non-relativistic string from the action

$$S = \int dt \int dx \left( \frac{1}{2} \mu_0 \left( \frac{\partial y}{\partial t} \right)^2 - \frac{1}{2} T_0 \left( \frac{\partial y}{\partial x} \right)^2 \right),$$

which is a particular case of the more general action  $S = \int dt dx \mathcal{L}(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x})$ . We can then define the momentum density and corresponding spatial quantity

$$\mathcal{P}^t = \frac{\partial \mathcal{L}}{\partial \dot{y}}, \quad \mathcal{P}^x = \frac{\partial \mathcal{L}}{\partial y'}.$$

The variation of the action is

$$\delta S = \int dt dx [\mathcal{P}^t \delta \dot{y} + \mathcal{P}^x \delta y'] = - \int dt dx \left[ \frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} \right] \delta y + \text{bndy terms}$$

and the action is made stationary,  $\delta S = 0$ , if the boundary terms vanish and if

$$\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0,$$

which when applied to the above particular choice of action gives the usual wave equation. The boundary terms must also be set to zero, and they involve  $\mathcal{P}^t \delta y$  at the time endpoints and  $\mathcal{P}^x \delta y$  at the space endpoints. Neumann BCs is to set  $\mathcal{P}^x = 0$  at the spatial endpoints (for all  $t$ ), and Dirichlet BCs is to set  $\delta y = 0$  (and thus  $\mathcal{P}^t = 0$ ) at the spatial endpoints.

Summary: string action:  $S = \int dt dx \mathcal{L}(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x})$ , with momentum densities

$$\mathcal{P}^t = \frac{\partial \mathcal{L}}{\partial \dot{y}}, \quad \mathcal{P}^x = \frac{\partial \mathcal{L}}{\partial y'}.$$

Least action gives the equations of motion

$$\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0.$$

The non-relativistic string has  $\mathcal{L} = \frac{1}{2} \mu_0 \dot{y}^2 - \frac{1}{2} T_0 y'^2$ , which we're going to replace with a relativistic version. For guidance, noted that a relativistic point particle of mass  $m$  has  $S = -mc \int ds = -mc^2 \int dt \sqrt{1 - v^2/c^2}$  and noted its reparametrization invariance: write  $x_\mu(\tau)$ , and can change worldline parameter  $\tau$  to an arbitrary new parameterization

$\tau'(\tau)$ , and the action is invariant. To see this use  $S = -mc \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}$  and change  $\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'} \frac{d\tau'}{d\tau}$  and note that  $S \rightarrow S$ . Euler Lagrange equations of motion:  $\frac{dp_\mu}{d\tau} = 0$ .

• As we discussed before, the action for a relativistic point particle of mass  $m$  is  $S = -mc \int ds = -mc^2 \int dt \sqrt{1 - v^2/c^2}$ . This gives  $\vec{p} = \partial_{\vec{v}} = \gamma m \vec{v}$  and  $H = \vec{p} \cdot \vec{v} - L = \gamma mc^2$ , both of which are constants of the motion (thanks to the time and spatial translation invariance). This has reparametrization invariance: write  $x_\mu(\tau)$ , and can change worldline parameter  $\tau$  to an arbitrary new parameterization  $\tau'(\tau)$ , and the action is invariant. To see this use  $S = -mc \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}$  and change  $\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'} \frac{d\tau'}{d\tau}$  and note that  $S \rightarrow S$ . The Euler Lagrange equations of motion are  $\frac{dp_\mu}{d\tau} = 0$ . When the particle is charged and in the presence of electric and magnetic fields, there is the new term in the action

$$S = \int (-mcds + \frac{q}{c} A_\mu dx^\mu), \quad (1)$$

which is manifestly relativistically invariant (and also reparameterization invariant). Note also that, under a gauge transformation, we have  $S \rightarrow S + \frac{qf}{c}$ , which does not affect the equations of motion (just as changing the Lagrangian by a total time derivative does not).

The lagrangian is thus  $L = -mc\sqrt{1 - \vec{v}^2/c^2} + \frac{q}{c} \vec{v} \cdot \vec{A} - q\phi$ . The momentum conjugate to  $\vec{r}$  is  $\vec{P} = \partial L / \partial \vec{v} = m\vec{v} / \sqrt{1 - \vec{v}^2/c^2} + \frac{q}{c} \vec{A}$ . The Hamiltonian is  $H = \vec{v} \cdot \vec{P} - L = \sqrt{m^2 c^4 + c^2 (\vec{P} - \frac{q}{c} \vec{A})^2} + q\phi$ . The equations of motion can be written as  $\frac{d^2 x^\mu}{d\tau^2} = \frac{q}{mc} F_{\mu\nu} \frac{dx^\nu}{d\tau}$ . In the non-relativistic limit we have  $H = \frac{1}{2m} (\vec{P} - \frac{q}{c} \vec{A})^2 + q\phi$ , where  $\vec{P} - \frac{q}{c} \vec{A} = m\vec{v}$ .

Recap:  $S = -mc \int ds + \frac{q}{c} \int A_\mu dx^\mu$  for a relativistic point particle, where we can write  $ds = \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\tau$ , with  $\dot{\cdot} \equiv \frac{d}{d\tau}$ , and  $\tau$  is the arbitrary worldline parameter, with reparameterization symmetry  $\tau \rightarrow \tau'$ .

• For a string world-sheet, we need two parameters,  $\xi^a$ ,  $a = 1, 2$ . The string trajectory is  $x : \Sigma \rightarrow M$ , where  $\Sigma$  is the 2d world-sheet, with local coordinates  $\xi^a$ , and  $M$  is the target space, with local coordinates  $x^\mu$ . The worldsheet area element is  $A = \int d^2 \xi \sqrt{|h|}$ , where  $h_{ab}$  is the worldsheet metric, and  $|h|$  is its determinant. Suppose that the target space has metric  $g_{\mu\nu}$ , with space-time length e.g.  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ . By writing  $dx^\mu = \partial_a x^\mu d\xi^a$ , we get

$$ds^2 = g_{\mu\nu} \frac{dx^\mu}{d\xi^a} \frac{dx^\nu}{d\xi^b} d\xi^a d\xi^b, \quad \text{so} \quad h_{ab} = g_{\mu\nu} \frac{dx^\mu}{d\xi^a} \frac{dx^\nu}{d\xi^b},$$

where this  $h_{ab}$  is called the induced metric. So the worldsheet area functional is

$$A = \int d^2 \xi \sqrt{\det(g_{\mu\nu} \frac{dx^\mu}{d\xi^a} \frac{dx^\nu}{d\xi^b})}.$$

ended here

• For strings in Minkowski spacetime, we write it instead as  $X^\mu(\tau, \sigma)$ . There is also a needed minus sign, as the area element is  $\sqrt{|g|}$ , actually involves the absolute value of the determinant, and the determinant is negative (just like  $\det \eta = -1$ ). So

$$A = \int d\tau d\sigma \sqrt{\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2 \left(\frac{\partial X}{\partial \sigma}\right)^2},$$

where the spacetime indices are contracted with the metric  $g_{\mu\nu}$ . To get an action with  $[S] = ML^2/T$ , we have

$$S_{Nambu-Goto} = -\frac{T_0}{c} \int_{\tau_i}^{\tau_f} d\tau \int d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2},$$

where we define  $\dot{X}^\mu \equiv \frac{dx^\mu}{d\tau}$  and  $X^{\mu'} \equiv \frac{\partial X^\mu}{\partial \sigma}$  and  $T_0$  is the string tension, with  $[T_0] = [F] = [ML/T^2]$ .

The action is reparameterization invariant: can take  $(\tau, \sigma) \rightarrow (\tau'(\tau, \sigma), \sigma'(\tau, \sigma))$  and get  $S \rightarrow S$ . Enormous symmetry/redundancy in choice of  $(\tau, \sigma)$ ; can “fix the gauge” to some convenient choice, and the physics is completely independent of the choice.