4/19/18 Lecture outline

$\star$ Reading: Zwiebach chapters 3,4,5.

• Last time: what about gravity in other D? In 4d, we have gravitational potential given by $\Phi_g^{(4)} = -GM/r$, which solves $\nabla^2\Phi_g^{(D)} = 4\pi G^{(D)}\rho_m$.

In any spacetime dimension, take $F_g = -m\nabla\Phi_g$, and $\nabla^2\Phi_g^{(D)} = 4\pi G^{(D)}\rho_m$. So in $\hbar = c = 1$ units, get $[F] = 2$, $[\Phi] = 0$, $[\rho] = D$, so $[G] = 2 - D$. Take $G = \ell_P^{D-2}$ in D spacetime dimensions.

Get $G^D = GV_C$, where V_C is the compactification volume. Example: consider a string of tension T that is wrapped on a circle in a 5th dimension, of radius ℓ_C. So $\rho_{5d} = T\delta(x_1)\delta(x_2)\delta(x_3)$ and $\rho_{4d} = \ell\rho_{5d}$, i.e. the 4d mass is $M = T\ell_c$, and the potential is $\Phi = -G^4M/r = -G^4T\ell_C/r$. Fits with $G^5 = G^4\ell_C$.

More generally, if we dimensionally reduce, then $G^{-1}_{reduced} = G^{-1}_{original}V_C$.

• Gravity is described, according to Einstein, by taking $ds^2 = g_{\mu\nu}dx^\mu dx^\mu$, with $g_{\mu\nu}$ dynamical and analogous to A_μ. The analog of $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the Einstein Hilbert action, $S \supset \int d^Dx\sqrt{|g|}\frac{1}{16\pi G}R$, where R is called the Ricci scalar curvature. We will not discuss it in detail, and the only point that I'd like to make for the moment is that it is convenient to take mass dimensions such that $[x] = -1$ and $[g_{\mu\nu}] = 0$ and then, since R is built from second derivatives of $g_{\mu\nu}$, it has $[R] = 2$, so $[G^{-1}] = D - 2$, i.e. $G \sim \ell_P^{D-2} \sim M_P^{2-D}$. This fits with the force between two masses being $F \sim Gm_1m_2/r^{D-2}$.

If we dimensionally reduce, get $G^{-1}_{reduced} = G^{-1}_{original}V$, as above.

• Return to the Lagrangian of electromagnetism

$$S = \int d^Dx\mathcal{L}, \qquad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{c}A_\mu j^\mu.$$

The two Maxwell's equations expressing absence of magnetic monopoles are, again, solved by setting $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$. The other two Maxwell's equations then come from the Euler -Lagrange equations of the above action upon varying $A_\mu \to A_\mu + \delta A_\mu$: the action is stationary when

$$\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)} - \frac{\partial\mathcal{L}}{\partial A_\mu} = 0 \to \partial_\nu F^{\mu\nu} = \frac{1}{c}j^\mu$$

• Nonrelativistic strings. $[T_0] = [F] = [E]/L = [\mu_0][v^2]$. Indeed, considering $F = ma$ for an element dx of the string yields the string wave equation $\frac{\partial^2y}{\partial x^2} - \frac{1}{v_0^2}\frac{\partial^2y}{\partial t^2} = 0$, with $v_0 = \sqrt{T_0/\mu_0}$. Endpoints at $x = 0$ and $x = a$. Can choose Dirichlet or Neumann BCs at these points. With Dirichlet at each end, $y_n(x) = A_n\sin(n\pi x/a)$ and the general solution is $y(x,t) = \sum_n y_n(x) \cos\omega_n t$, where $\omega_n = v_0 n\pi/a$ (and the A_n are determined from the initial conditions, by Fourier transform).

ended here

The nonrelativistic string action is $S = \int dtL$ where L is the kinetic energy minus potential energy, which gives

$$S = \int dt \int dx \left(\tfrac{1}{2}\mu_0 (\frac{\partial y}{\partial t})^2 - \tfrac{1}{2} T_0 (\frac{\partial y}{\partial x})^2 \right),$$

which is a particular case of the more general action $S = \int dtdx\mathcal{L}(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x})$. We can then define the momentum density and corresponding spatial quantity

$$\mathcal{P}^t = \frac{\partial \mathcal{L}}{\partial \dot{y}}, \qquad \mathcal{P}^x = \frac{\partial \mathcal{L}}{\partial y'}.$$

The variation of the action is

$$\delta S = \int dtdx[\mathcal{P}^t \delta\dot{y} + \mathcal{P}^x \delta y'] = -\int dtdx[\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x}]\delta y + \text{bndy terms}$$

and the action is made stationary, $\delta S = 0$, if the boundary terms vanish and if

$$\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0,$$

which when applied to the above particular choice of action gives the usual wave equation. The boundary terms must also be set to zero, and they involve $\mathcal{P}^t \delta y$ at the time endpoints and $\mathcal{P}^x \delta y$ at the space endpoints. Neumann BCs is to set $\mathcal{P}^x = 0$ at the spatial endpoints (for all t), and Dirichlet BCs is to set $\delta y = 0$ (and thus $\mathcal{P}^t = 0$) at the spatial endpoints.

• The action for a relativistic point particle of mass m is $S = -mc\int ds = -mc^2 \int dt\sqrt{1 - v^2/c^2}$. This gives $\vec{p} = \partial_{\vec{v}} = \gamma m\vec{v}$ and $H = \vec{p} \cdot \vec{v} - L = \gamma mc^2$, both of which are constants of the motion (thanks to the time and spatial translation invariance).

• Reparametrization invariance: write $x_\mu(\tau)$, and can change worldline parameter τ to an arbitrary new parameterization $\tau'(\tau)$, and the action is invariant. To see this use $S = -mc\int \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$ and change $\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau}$ and note that $S \to S$. The Euler Lagrange equations of motion are $\frac{dp_\mu}{d\tau} = 0$.