4/12/18 Lecture outline

⋆ Reading: Zwiebach chapter 3.

• Use units where Maxwell's equations are $\nabla \times \vec{E} = -\frac{1}{c}\partial_t \vec{B}$, $\nabla \cdot \vec{B} = 0$, $\nabla \cdot \vec{E} = \rho$, $\nabla \times \vec{B} = \frac{1}{c}\vec{j} + \frac{1}{c}\partial_t \vec{E}$. The first two equations can be solved by introducing the scalar and vector potential: $\vec{B} = \nabla \times \vec{A}$, $\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \nabla\phi$. Gauge invariance: all physics (including $\vec{E}$ and $\vec{B}$) invariant under

$$\phi \to \phi - \frac{1}{c}\frac{\partial f}{\partial t}, \qquad \vec{A} \to \vec{A} + \nabla f, \tag{1}$$

for an arbitrary function $f(t, \vec{x})$. This initially dull sounding invariance takes a fundamental role in modern high energy physics: such local (because f can vary locally over space-time) gauge symmetries are in direct correspondence with forces!

• Maxwell's equations in relativistic form. Like last time, $x^\mu = (ct, \vec{x})$ and also use $\partial_\mu = (c\partial_t, \nabla)$ (and thus $\partial^\mu = (-c\partial_t, \nabla)$). $\vec{E}$ and $\vec{B}$ combine into an antisymmetric, 2-index, 4-tensor $F_{\mu\nu} = -F_{\nu\mu}$, via $F_{0i} = -E_i$ and $F_{ij} = \epsilon_{ijk}B^k$, i.e.

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}.$$

As usual, we can raise and lower indices with $\eta_{\mu\nu}$, e.g. $F^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\sigma}F_{\lambda\sigma}$ and with the book's sign convention this gives a minus sign each time a time component is raised or lowered. So $F^{0i} = -F_{0i}$ and $F^{ij} = F_{ij}$, where i and j refer to the spatial components, i.e. the matrix $F^{\mu\nu}$ is similar to that above, but with $\vec{E} \to -\vec{E}$.

Under Lorentz transformations, $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$, the electric and magnetic fields transform as $F^{\mu'\nu'} = \Lambda^{\mu'}{}_\sigma \Lambda^{\nu'}{}_\rho F^{\sigma\rho}$. Sources combine into a 4-vector as $j^\mu = (c\rho, \vec{j})$, and charge conservation is the Lorentz-invariant equation $\partial_\mu j^\mu = 0$.

The Lorentz force law is $f^\mu_{E\&M} = \pm q F^{\mu\nu}u_\nu$ (go through the exercise of checking the sign on the board). Maxwell's equations in relativistic form are $\partial_{[\mu}F_{\rho\sigma]} = 0$, and $\partial_\lambda F^{\mu\lambda} = \frac{1}{c}j^\mu$ (this convention, with indices not next to each other contracted, is peculiar to the $(-+++)$ choice of $\eta_{\mu\nu}$), which exhibits that they transform covariantly under Lorentz transformations.

The scalar and vector potential combine to the 4-vector $A^\mu = (\phi, \vec{A})$ and the first two Maxwell equations are solved via $F^{\mu\nu} = \partial^{[\mu}A^{\nu]}$. The gauge invariance is $A^\mu \to A^\mu + \partial^\mu f$. E.g. Lorentz gauge: $\partial_\mu A^\mu = 0$. Physics is independent of choice of gauge, but some are sometimes more convenient than others along the way, depending on what's being done. In Lorentz gauge, the remaining Maxwell equations are $\partial_\mu \partial^\mu A^\nu = -\frac{1}{c} j^\nu$ (still some gauge freedom). In empty space we set $j^\mu = 0$ and the plane wave solutions are $A^\mu = \epsilon^\mu(p) e^{ip\cdot x}$, where $p^2 = 0$ (massless) and $p \cdot \epsilon = 0$. Can still shift $\epsilon^\mu \to \epsilon^\mu + \alpha p^\mu$, so 2 independent photon polarizations ϵ^μ.

• The action for a relativistic point particle of mass m is $S = -mc \int ds = -mc^2 \int dt \sqrt{1 - v^2/c^2}$. This gives $\vec{p} = \partial_{\vec{v}} = \gamma m \vec{v}$ and $H = \vec{p} \cdot \vec{v} - L = \gamma mc^2$, both of which are constants of the motion (thanks to the time and spatial translation invariance).

When the particle is charged and in the presence of electric and magnetic fields, there is the new term in the action

$$S = \int (-mcds + \frac{q}{c} A_\mu dx^\mu), \tag{2}$$

which is manifestly relativistically invariant (and also repparameterization) invariant. Note also that, under a gauge transformation, we have $S \to S + \frac{qf}{c}$, which does not affect the equations of motion (just as changing the Lagrangian by a total time derivative does not).

The lagrangian is thus $L = -mc\sqrt{1 - \vec{v}^2/c^2} + \frac{q}{c} \vec{v} \cdot \vec{A} - q\phi$. The momentum conjugate to $\vec{r}$ is $\vec{P} = \partial L/\partial \vec{v} = m\vec{v}/\sqrt{1 - \vec{v}^2/c^2} + \frac{q}{c}\vec{A}$. The Hamiltonian is $H = \vec{v} \cdot \vec{P} - L = \sqrt{m^2c^4 + c^2(\vec{P} - \frac{q}{c}\vec{A})^2} + q\phi$. The equations of motion can be written as $\frac{d^2 x^\mu}{d\tau^2} = \frac{q}{mc} F_{\mu\nu} \frac{dx^\nu}{d\tau}$. In the non-relativistic limit we have $H = \frac{1}{2m}(\vec{P} - \frac{q}{c}\vec{A})^2 + q\phi$, where $\vec{P} - \frac{q}{c}\vec{A} = m\vec{v}$.

• In QM, gauge transformation $A^\mu \to A^\mu + \partial^\mu f$ accompanies giving an overall, local phase to the QM wavefunction $\psi \to e^{iqf/\hbar c}\psi$, where q is the electric charge of the field.