4/10/18 Lecture outline

⋆ Reading: Zwiebach chapters 1 and 2.

• Last time: Statement of relativity principle: physics is indistinguishable among all inertial frames. If one frame is inertial, all frames moving relative to that at a constant velocity are also inertial. Related by $x^{\mu'} = \Lambda^{\mu'}_{\nu} x^\nu$ where $\Lambda^T \eta \Lambda = \eta$, where $\eta$ is the metric, chosen to have mostly plus convention. All 4-vectors transform the same way: $b^{\mu'} = \Lambda^{\mu'}_{\nu} b^\nu$. A tensor transforms as $T^{\mu'\nu'} = \Lambda^{\mu'}_{\kappa} \Lambda^{\nu'}_{\sigma} T^{\kappa\sigma}$. Can contract or raise or lower indices with $\eta_{\mu\nu}$, e.g. $a^\mu b^\nu \eta_{\mu\nu} = -a^0 b^0 + \vec{a} \cdot \vec{b}$ and $T^\mu_\mu = -T^{00} + \delta_{ij} T^{ij}$ are 4scalars. 4-scalars are invariant under Lorentz transformation. The proper time element $d\tau = \sqrt{-dx^\mu dx_\mu / c^2}$ is a 4-scalar. $u^\mu = dx^\mu / d\tau$ is the velocity 4-vector, and $d^2 x^\mu / d\tau^2$ is the acceleration 4-vector. The statement of relativity requires that all formulas relate quantities that transform the same way, for example $\vec{F} = m\vec{a}$ can fit with relativity if it becomes a 4-vector equation: $f^\mu = m dp^\mu / d\tau$ (the time component here relates power to the time derivative of energy).

$p^\mu = (E/c, p_x, p_y, p_z)$, with $p_\mu p^\mu = -m^2 c^2$. $p^\mu$ transforms as a Lorentz 4-vector, $p^{\mu'} = \Lambda^{\mu'}_{\nu} p^\nu$. Proper time: $ds^2 = c^2 dt_p^2 = c^2 dt^2 (1 - \beta^2)$. $u^\mu = c dx^\mu / ds = dx^\mu / dt_p = \gamma(c, \vec{v})$, and $u_\mu u^\mu = -c^2$. A massive point particle has $p^\mu = m u^\mu$. Massless particles, like the photon, have $p^\mu$ with $p^\mu p_\mu = 0$.

• Quantum mechanics: replace $p^\mu = (H/c, \vec{p}) \to -i\hbar \partial^\mu$. Free particle wavefunction $\psi \sim \exp(ip \cdot x / \hbar)$; $p_\mu x^\mu \equiv p \cdot x$ is Lorentz invariant.

• Light cone coordinates: $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$. The bad: spoils rotational symmetry. The good: will make it easier to quantize string theory (avoids having to discuss here more advanced alternatives, which do not require going to the light cone coordinates). $-ds^2 = -2dx^+ dx^- + dx_2^2 + dx_3^2 = -\widehat{\eta}_{\mu\nu} dx^\mu dx^\nu$. $a_\pm = -a^\mp$. Take $p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^1) = -p_\mp$. So $i\hbar \partial_{x^+} \to -p_+ = E_{lc}/c$, i.e. $p^- = E_{lc}/c$.

• Extra (spacelike) dimensions, e.g. 2 extra dimensions: $-ds^2 = -c^2 dt^2 + \sum_{i=1}^5 (dx^i)^2$. Consider one extra space dimension, taken to be a circle, $x \sim x + 2\pi R$. Now consider $(x, y) \sim (x + 2\pi R, y) \sim (x, y + 2\pi R)$; gives a torus. Orbifold, e.g. $z \sim e^{i\pi i/N} z$, gives a cone (singular at fixed point).

• Recall QM: $[x^i, p_j] = i\hbar \delta^i_j$. Particle in square well box of size $a$: $E = (n\pi/a)^2 / 2m$. Now particle in periodic box, $x_4 \sim x_4 + 2\pi R$. The other directions, $x^\mu$, are given by some standard Hamiltonian, e.g. the hydrogen atom, which we'll call $H_{4d}$. So $H_{5d} = H_{4d} + \widehat{p}_4^2 / 2m$, with $\widehat{p}_4 = -i\hbar \partial_{x_4}$ in position space. The 4d energy eigenstates are then given by separation of variables to be $\psi_{E_{5d}}(\vec{x}, x_4) = \psi_{E_{4d}}(\vec{x}) \frac{1}{\sqrt{2\pi R}} e^{i\ell x_4 / R}$, with $\ell$ an integer, and $\psi_{E_{4d}}$ is an energy eigenstate of the 4d problem. So $E_{5d} = E_{4d} + \ell^2 / 2mR^2$. For $R$ small, the low energy states are simply those with $\ell = 0$, and the extra dimension is unseen.