6/7/18 Lecture outline

$\star$ Reading: Zwiebach chapter 12.

• Last time: we completed the quantization of the open string, finding $D = 26$ and $H = 2\alpha' p^+ p^- = L_0^\perp - 1$, and the states are

$$|\lambda\rangle = \prod_{n=1}^{\infty} \prod_{I=2}^{25} (a_n^{I\dagger})^{\lambda_{n,I}} |p^+, \vec{p}_T\rangle.$$

These states are eigenstates of

$$M^2 = \frac{1}{\alpha'}(-1 + N^\perp), \qquad N^\perp \equiv \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I,$$

with eigenvalues

$$M^2 = \frac{1}{\alpha'}(-1 + N^\perp), \qquad N^\perp = \sum_n \sum_I n\lambda_{n,I}.$$

The groundstate is tachyonic (!). The first excited state is a massless spacetime vector with $D-2$ polarizations, i.e. a massless gauge field, like the photon (but in $D = 26$)!

The tachyon is related to the fact that the D25 brane is unstable, it decays to the closed string vacuum. The closed bosonic string is also unstable, as we'll see next time. These instabilities can be cured by adding fermions and considering the superstring. Then the critical spacetime dimension is $D = 10$.

The eigenstates satisfy the worldsheet SE:

$$i\frac{\partial}{\partial \tau}|\lambda\rangle = H|\lambda\rangle = (L_0^\perp - 1)|\lambda\rangle.$$

Writing $x^+ = 2\alpha' p^+ \tau$, this becomes

$$(i\frac{\partial}{\partial x^+} - \frac{1}{2p^+}(p^I p^I + M^2))\phi_\lambda(x^+, p^+, \tau),$$

which is the KG (or generalization) wave equation for the corresponding field in spacetime.

• Now consider closed string case. Recall gauge conditions $n \cdot X = \alpha'(n \cdot p)\tau$, $n \cdot p = 2\pi n \cdot \mathcal{P}^\tau$, which yielded the constraints $(\dot{X} \pm X')^2 = 0$ and then the EOM were simply $(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0$. For the closed string, this means that $X^\mu(\tau, \sigma) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma)$. The general solutions can then be written as

$$X_R^\mu(v) = \tfrac{1}{2}x_0^{L\mu} + \sqrt{\tfrac{1}{2}\alpha'}\alpha_0^\mu v + i\sqrt{\tfrac{1}{2}\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-inv},$$

and a similar expression for X_L^μ, with modes $\widetilde{\alpha}_n^\mu$. Since $X^\mu(\tau,\sigma+2\pi)=X^\mu(\tau,\sigma)$, $\widetilde{\alpha}_0^\mu=\alpha_0^\mu$. Computing $\mathcal{P}^{\mu\mu}=\dot{X}^\mu/2\pi\alpha'$ then yields $\alpha_0^\mu=\sqrt{\frac{1}{2}\alpha'}p^\mu$.

The theory is quantized by taking $[X^I(\tau,\sigma),\mathcal{P}^{\tau J}(\tau,\sigma')]=i\delta(\sigma-\sigma')\eta^{IJ}$, which implies that

$$[\alpha_m^I,\alpha_n^J]=m\delta_{m+n,0}\eta^{IJ},\qquad [\widetilde{\alpha}_m^I,\widetilde{\alpha}_n^J]=m\delta_{m+n,0}\eta^{IJ}$$

with no commutator between the left and right movers. It's now very similar to the open string case, but with the two sets of decoupled oscillators for the left and right movers. We define

$$(\dot{X}^I+X'^I)^2\equiv 4\alpha'\sum_n \widetilde{L}_n^\perp e^{-in(\tau+\sigma)},$$

and a similar expansion for $(\dot{X}^I-X'^I)^2$ and $L_n^\perp$, involving $\tau-\sigma$. Then $L_n^\perp=\frac{1}{2}\sum_p\alpha_p^I\alpha_{n-p}^I$, and $L_0^\perp=\frac{\alpha'}{4}p^Ip^I+N^\perp$. The X^- are given in terms of these much as in the open string case, $\sqrt{2\alpha'}\alpha_n^-=2L_n^\perp/p^+$, with a similar expression for the left movers. The worldsheet Hamiltonian is $H=L_0^\perp+\widetilde{L}_0^\perp-2$ and $M^2=-p^2=2p^+p^--p^Ip^I=\frac{2}{\alpha'}(N^\perp+\widetilde{N}^\perp-2)$.

The closed string states are given by acting with left and right moving creation operators on $|p^+,p^I\rangle$, with the constraint that $N^\perp=\widetilde{N}^\perp$ (because of translation symmetry in shifting σ). In summary, the spectrum of states is given by

$$\begin{gathered}|\lambda,\widetilde{\lambda}\rangle=[\prod_{n=1}^\infty\prod_{I=2}^{D-1}(a_n^{I\dagger})^{\lambda_{n,I}}][\prod_{n=1}^\infty\prod_{I=2}^{D-1}(\widetilde{a}_n^{I\dagger})^{\widetilde{\lambda}_{n,I}}]|p^\mu\rangle\\ M^2=-p^2=2(N_\perp+\widetilde{N}_\perp-2)/\alpha',\quad N^\perp=\sum_{n=1}^\infty\sum_{I=1}^{D-1}n\lambda_{n,I},\quad N^\perp=\sum_{n=1}^\infty\sum_{I=1}^{D-1}n\widetilde{\lambda}_{n,I},\end{gathered}\tag{1}$$

where there is a requirement that $N^\perp=\widetilde{N}^\perp$ to have σ translation invariance.

The state with $N^\perp=0$ is the bosonic closed string tachyon. Those with $N^\perp=1$ are given by a $(D-2)^2$ matrix of indices in the transverse directions, and these are massless. The symmetric traceless part is the graviton, the antisymmetric tensor is a gauge field $B_{\mu\nu}$ which is an analog of A_μ, and the trace part is ϕ, called the "dilaton."

- Let's count the states by defining $f(x)=\mathrm{Tr}_{states}x^{\alpha' M^2}$. Find

$$f_{os}(x)=x^{-1}\prod_{n=1}^\infty\frac{1}{(1-x^n)^{24}}$$

where we set $D-2=24$. Similarly, for the closed string case, we have

$$f_{closed}(x,\bar{x})=f_{os}(x)f_{os}(\bar{x}),$$

where we need to project out those states with different powers of x and $\bar{x}$.

• Consider the closed, bosonic string on a circle, $X_{25} \sim X_{25} + 2\pi R$. If we were dealing with particles rather than strings, we know what would happen: the momentum in the circle direction is quantized (by $\psi \sim e^{ip\cdot x}$ being set equal to itself when going around the circle) as

$$p_{25} = \frac{n}{R}, \qquad n = 0, \pm 1, \pm 2 \ldots.$$

For a big circle, these are closely spaced together, and for a small circle they are widely separated. That's why it's hard to experimentally rule out the absence of tiny, rolled up, extra dimensions: it could just take more energy than we can make presently to excite one of the $n \neq 0$ "Kaluza-Klein modes."

Now we're going to describe something bizarre about strings: there is a symmetry, called T-dualtiy, which makes the physics invariant under $R \leftrightarrow \alpha'/R$. This is strange: a very big circle is physically indistinguishable from a very small circle! The reason is that, in addition to momentum, there are string winding modes, and T-duality exchanges them. For a big circle, the momentum modes are light and the winding modes are heavy, and for a tiny circle they're reversed, but same physics. Smallest possible effective distance, $R = \sqrt{\alpha'}$.

The winding number is given by $X(\tau, \sigma + 2\pi) - X(\tau, \sigma) = m(2\pi R)$. We then have $X = X_L + X_R$ with

$$X_L(\tau + \sigma) = const. + \tfrac{1}{2}\alpha'(p + w)(\tau + \sigma) + \text{oscillators},$$

$$X_R(\tau - \sigma) = const + \tfrac{1}{2}\alpha'(p - w)(\tau - \sigma) + \text{oscillators}.$$

Here

$$p = \frac{n}{R}, \qquad w = \frac{mR}{\alpha'}.$$

The T-duality symmetry comes from the symmetry $(p_L, p_R) \to (p_L, -p_R)$, where

$$p_L = \frac{n}{R} + \frac{mR}{\alpha'}, \qquad p_R = \frac{n}{R} - \frac{mR}{\alpha'}.$$

Also, to have $X(\tau, \sigma + 2\pi) \sim X(\tau, \sigma) + 2\pi R m$, we need $N^{\perp} - \widetilde{N}^{\perp} = nm$.

• Let's now consider some aspects of string thermodynamics. The number of string states grows very rapidly with excitation number, and it turns out that this puts an upper limit on the temperature, beyond which the partition function would not converge.

Before getting into thermodynamics, let's count string states. Recall that we counted states of the open bosonic string via

$$f_{os}(x) = \mathrm{Tr}_{states} x^{\alpha' M^2} = \left(\frac{1}{x^{1/24} \prod_{n=1}^{\infty}(1-x^n)} \right)^{(D-2)} \equiv \sum_{N=0}^{\infty} p_{D-2}(N) x^{N-(D-2)/24}.$$

We saw that $D = 26$, but let's keep it as a parameter for the moment. Here $p_{D-2}(N)$ is the number of distinct partitions of N into arbitrary numbers of non-negative integers, each of which can have $D-2$ labels. This corresponds to how many choices of $\lambda_{I,n}$ there are such that $\prod_{I=2}^{D-1} \prod_{n=1}^{\infty} (a_n^{I\dagger})^{\lambda_{I,n}}$ has $N^\perp = \sum_n \sum_I n\lambda_{I,n} = N$. Let's consider $p_1(N) = p(N)$ as an illustration: $p(5) = 7$, $p(10) = 42$, find $p(N)$ grows rapidly with N. The large N behavior of $p(N)$ was studied long ago by number theorists Hardy and Ramanujan:

$$p(N \gg 1) \approx \frac{1}{4N\sqrt{3}} \exp(2\pi \sqrt{\frac{N}{6}}).$$

Can also show:

$$p_b(N \gg 1) \approx \frac{1}{\sqrt{2}} \left(\frac{b}{24} \right)^{(b+1)/2} N^{-(b+3)/4} \exp(2\pi \sqrt{\frac{Nb}{6}}).$$

Note the appearance of 24 in this number theory formula, which will be related to $D-2 = 24$ in string theory. In fact, these formula were derived by relating the above generating functions to the Dedekind eta function:

$$\eta(\tau) \equiv e^{i\pi\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}).$$

Note that this function, defined long ago by mathematicians, nicely allows us to write

$$f_{os}(x = e^{2\pi i \tau}) = \eta(\tau)^{-(D-2)}.$$

An important property of the eta function (both for math, and for string theory!) is

$$\eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau),$$

and this allows us to relate the $x \to 1$ limit, which is relevant for extracting $p(N \gg 1)$, to another limit: $x \to 1$ is $\tau \to 0i$, which can be related to $-1/\tau \to i\infty$.

Setting $\tau = i\tau_2$, the above generating functions start to resemble partition functions, with $H = \alpha' M^2$. (Actually, they are partition functions, but on the worldsheet for the moment.)

$$f_{os} = \mathrm{Tr}_{states} e^{-2\pi\tau_2\alpha' M^2} = \sum_{N=0}^{\infty} p_{D-2}(N) e^{-2\pi\tau_2(N-1)}.$$

For large N, we have $M^2 \approx N/\alpha'$ and, taking $E = M$, we have $\sqrt{N} = \sqrt{\alpha'}E$. The entropy of string states with energy E is then

$$S(E) = k \ln \Omega(E) = k \ln p_{24}(N = \sqrt{\alpha' E}) \approx k 4\pi \sqrt{\alpha'} E.$$

Then

$$\frac{1}{kT} = \frac{1}{k}\frac{\partial S}{\partial E} = 4\pi\sqrt{\alpha'} \equiv \frac{1}{kT_H},$$

where T_H is the Hagedorn temperature.

To fully compute the spacetime partition function, we must write $E = \sqrt{\vec{p}^2 + M^2}$ and do the integral over momentum, $V \int d^{D-1}p/(2\pi\hbar)^{D-1}$. Find that $Z_{string}(T)$ has a pole as $T \to T_H$: $Z_{string} \sim C/(T - T_H)$, with C a constant.

• Superstrings! The bosonic string has fields $X^I(\tau, \sigma)$, which are $D-2$ worldsheet scalars. Now we introduce $D-2$ worldsheet fermions

$$\Psi_R(\tau - \sigma)^I, \qquad \Psi_L^I(\tau + \sigma).$$

Here R and L are for right and left moving, and $I = 2 \ldots D-2$ (spacetime vector indices). The light cone gauge action is

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \left(\dot{X}^I \dot{X}^I - X^{I'} X^{I'} + \Psi_R^I (\partial_\tau + \partial_\sigma) \Psi_R^I + \Psi_L^I (\partial_\tau - \partial_\sigma) \Psi_L^I \right).$$

Note that the bosons X^I have the usual quadratic in derivatives terms (like $L = \frac{1}{2} m \dot{x}^2$) whereas the fermions Ψ have linear in derivatives terms. The fermion and its action can roughly be thought of as the square-root of a boson and its action. The terms in the action above is the 2d worldsheet version of the Dirac equation action. The classical equations of motion from the Euler Lagrange equations are just $(\partial_\tau + \partial_\sigma)\Psi_R = 0$ and $(\partial_\tau - \partial_\sigma)\Psi_L = 0$, which are solved by $\Psi_R = \Psi_R(\tau - \sigma)$ and $\Psi_L = \Psi_L(\tau + \sigma)$.

There are two choices of boundary conditions for left movers, and similarly two choices for right movers:

$$\Psi^I(\tau, \sigma + 2\pi) = \pm \Psi^I(\tau, \sigma), \qquad + : \text{Ramond}, \quad - : \text{Nevu-Schwarz}.$$

In the NS sector we have

$$\Psi^I_{NS} \sim \sum_{n=-\infty}^{\infty} b^I_{n+\frac{1}{2}} e^{-i(n+\frac{1}{2})(\tau-\sigma)}.$$

In the R sector we have

$$\Psi^I_R \sim \sum_{n=-\infty}^{\infty} d^P_n e^{-in(\tau-\sigma)}.$$

In the NS sector we have

$$\Psi^I_{NS} \sim \sum_{n=-\infty}^{\infty} b^I_{n+\frac{1}{2}} e^{-i(n+\frac{1}{2})(\tau-\sigma)}.$$

In the R sector we have

$$\Psi^I_R \sim \sum_{n=-\infty}^{\infty} d^P_n e^{-in(\tau-\sigma)}.$$

The modes satisfy

$$\{b^I_r, b^J_s\} = \delta_{r+s,0}\delta^{IJ}, \qquad \{d^I_n, d^J_m\} = \delta_{n+m,0}\delta^{IJ},$$

where $\{A, B\} \equiv AB + BA$ is the anti-commutator, reflecting the fermionic nature of the modes.

The NS sector states are

$$|\lambda, \rho\rangle_{NS} = \prod_{I=2}^{D-2} (a^{I\dagger}_n)^{\lambda_{n,I}} \prod_{J=2}^{D-1} \prod_{r=\frac{1}{2},\frac{3}{2}\cdots} (b^J_{-r})^{\rho_{r,J}} |NS\rangle \otimes |p\rangle,$$

where the $\rho_{r,J}$ are either zero or one (Fermi statistics).

The R sector states are

$$|\lambda, \rho\rangle_R = \prod_{I=2}^{D-2} \prod_n (a^{I\dagger}_n)^{\lambda_{n,I}} \prod_{J=2}^{D-1} \prod_{m=1}^{\infty} (d^J_{-m})^{\rho_{m,J}} |R_A\rangle \otimes |p\rangle,$$

The upshot in this case is that $D = 10$ spacetime dimensions is needed. The mass-squared operator in the NS sector is

$$\alpha' M^2 = N^\perp + \tfrac{1}{2}(D-2)(-\frac{1}{12} - \frac{1}{24}),$$

where the $-1/12$ was seen in the bosonic case, and the $-1/24$ is the analog coming from reordering the b_r. As in the bosonic case, the commutator $[M^{-I}, M^{-J}] = 0$ determines the spacetime dimension, here to be $D = 10$. Similarly, in the R-sector, we have

$$\alpha' M^2 = N^{\perp}, \qquad N^{\perp} = \sum_{p=1}^{\infty} p a_p^{\dagger I} a_p^I + \sum_{m=1}^{\infty} m d_{-m}^I d_m^I.$$

The NS spectrum generating function is

$$f_{NS}(x) = \frac{1}{\sqrt{x}} \prod_{n=1}^{\infty} \left(\frac{1 + x^{n-\frac{1}{2}}}{1 - x^n} \right)^8 .$$

The R sector spectrum generating function is

$$f_{R\pm}(x) = 8 \prod_{n=1}^{\infty} \left(\frac{1 + x^n}{1 - x^n} \right)^8$$

where 8 accounts for the ground state degeneracy associated with d_0^I, in either the R_+ or the R_- sector. We should also GSO project the NS sector, i.e. throw away states with $(-1)^F = -1$ to get the $NS+$ states, with generating function

$$f_{NS+}(x) = \frac{1}{2\sqrt{x}} \left[\prod_{n=1}^{\infty} \left(\frac{1 + x^{n-\frac{1}{2}}}{1 - x^n} \right)^8 - \left(\frac{1 - x^{n-\frac{1}{2}}}{1 - x^n} \right)^8 \right] .$$

This projects out the tachyon – nice! Moreover, the states in $f_{R\pm}$ are spacetime fermions, whereas those in $f_{NS,+}$ are spacetime bosons, and their spectrum is degenerate, thanks to the identity $f_{R\pm}(x) = f_{NS+}(x)$ (which was proven as a mathematical identity around 150 years before the superstring was even first thought of!).

• For closed superstrings we can take the $NS+$ sector for both left and right movers, and the $R-$ sector for both left and right movers; this is the IIB superstring. Or we could take the $NS+$ sector for both left and right movers, and the $R-$ sector for left movers and the $R+$ sector for right movers; this is the IIA superstring.

The massless (NS+, NS+) states for both of these string theories consist of

$$\widetilde{b}^I_{-\frac{1}{2}} |NS\rangle_L \otimes b^J_{-\frac{1}{2}} |NS\rangle_R \otimes |p\rangle.$$

As in the bosonic case, these correspond to $g_{\mu\nu}$,, $B_{\mu\nu}$, ϕ.