

6/5/18 Lecture outline

★ Reading: Zwiebach chapter 12.

• Last time: Lorentz transformations correspond to the inf. transformations  $\delta x^\mu = \epsilon^{\mu\nu} x_\nu$ , and the corresponding conserved Noether charges are the generalized angular momenta  $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ . These generate rotations and boosts. Have e.g.  $[M^{\mu\nu}, x^\rho] = i\eta^{\mu\rho} x^\nu - i\eta^{\nu\rho} x^\mu$  and  $[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho} M^{\nu\sigma} \pm (\text{perms})$ .

In light cone coordinates, have  $M^{IJ}, M^{\pm I}, M^{+-}$ , with e.g.  $[M^{+-}, M^{+I}] = iM^{+I}$  and  $[M^{-I}, M^{-J}] = 0$ . There are some ordering issues for some of these, where operators which don't commute need to be replaced with symmetrized averages, e.g.  $M^{+-} = -\frac{1}{2}(x_0^- p^+ + p^+ x_0^-)$  and  $M^{-I} = x_0^- p^I - \frac{1}{2}(x_0^I p^- + p^- x_0^I)$ .

Open string. Imposed constraints  $(\dot{X} \pm X')^2 = 0$  to get

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X'^{\mu'}, \quad \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu.$$

In light cone gauge, much as with the point particle, the independent variables are  $(X^I, (\sigma)x_0^-, \mathcal{P}^{\tau I}(\sigma), p^+)$ . In the H picture the capitalized ones depend (implicitly) on  $\tau$  too. The commutation relations are

$$[X^I(\sigma), \mathcal{P}^{\tau J}(\sigma')] = i\eta^{IJ} \delta(\sigma - \sigma'), \quad [x_0^-, p^+] = -i.$$

The Hamiltonian is taken to be

$$H = 2\alpha' p^+ p^- = 2\alpha' p^+ \int_0^\pi d\sigma \mathcal{P}^{\tau-} = \pi\alpha' \int_0^\pi d\sigma (\mathcal{P}^{\tau I} \mathcal{P}^{\tau I} + X'^I X'^I (2\pi\alpha')^{-2})$$

Can write  $H = L_0^\perp$  since  $L_0^\perp = 2\alpha' p^+ p^-$ .

This  $H$  properly yields the expected time derivatives, e.g.  $\dot{X}^I = 2\pi\alpha' \mathcal{P}^{\tau I}$ .

Recall the solution with  $N$  BCs:

$$X^I(\tau, \sigma) = x_0^I + \sqrt{2\alpha'} \alpha_0^I \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^I \cos n\sigma e^{-in\tau}. \quad (1)$$

The needed commutators are ensured by

$$[\alpha_m^I, \alpha_n^J] = m\eta^{IJ} \delta_{n+m,0}.$$

Also, as before, we define  $\alpha_0^I \equiv \sqrt{2\alpha'} p^I$ . Now define  $\alpha_{n>0}^\mu = \sqrt{n} a_n^\mu$  and  $\alpha_{-n}^\mu = a_n^{\mu*} \sqrt{n}$  to rewrite the above as

$$[a_m^I, a_n^{J\dagger}] = \delta_{m,n} \eta^{IJ}. \quad (2)$$

- The transverse light cone coordinates can be described by

$$S_{l.c.} = \int d\tau d\sigma \frac{1}{4\pi\alpha'} (\dot{X}^I \dot{X}^I - X^{I'} X^{I'}).$$

Gives correct  $\mathcal{P}^{\tau I} = \partial\mathcal{L}/\partial\dot{X}^I$  and correct  $H = \int d\sigma(\mathcal{P}^{\tau I} \dot{X}^I - \mathcal{L})$ .

Writing  $X^I(\tau, \sigma) = q^I(\tau) + 2\sqrt{\alpha'} \sum_{n=1}^{\infty} q_n^I(\tau) n^{-1/2} \cos n\sigma$  and plugging into the action above gives

$$S = \int d\tau \left[ \frac{1}{4\alpha'} \dot{q}^I \dot{q}^I + \sum_{n=1}^{\infty} \left( \frac{1}{2n} \dot{q}_n^I \dot{q}_n^I - \frac{n}{2} q_n^I q_n^I \right) \right]$$

and

$$H = \alpha' p^I p^I + \sum_{n=1}^{\infty} \frac{n}{2} (p_n^I p_n^I + q_n^I q_n^I).$$

A bunch of harmonic oscillators. Relate to (1) and (2), showing that the  $a_m$  can be interpreted as the usual harmonic oscillator annihilation operators.

- Summary: we fix  $X^+$  to be simply related to  $\tau$ , find that the  $X^I$  are given by simple harmonic oscillators, and  $X^-$  is a complicated expression, fully determined in terms of the transverse direction quantities:

$X^+(\tau\sigma) = 2\alpha' p^+ \tau = \sqrt{2\alpha'} \alpha_0^+ \tau$ . For  $X^-$  recall expansion, with  $\sqrt{2\alpha'} \alpha_n^- = \frac{1}{p^+} L_n^\perp$ , where  $L_n^\perp \equiv \frac{1}{2} \sum_p \alpha_{n-p}^I \alpha_p^I$  is the transverse Virasoro operator. Recall  $[\alpha_m^I, \alpha_n^J] = m\delta^{IJ} \delta_{m+n,0}$ . There is an ordering ambiguity here, only for  $L_0^\perp$ :

$$L_0^\perp = \frac{1}{2} \alpha_0 \alpha_0 + \frac{1}{2} \sum_{p=1}^{\infty} \alpha_{-p}^I \alpha_p^I + \frac{1}{2} \sum_{p=1}^{\infty} \alpha_p^I \alpha_{-p}^I.$$

The ordering in the last terms need to be fixed, so the annihilation operator  $\alpha_p$  is on the right, using  $\alpha_p^I \alpha_{-p}^I = \alpha_{-p}^I \alpha_p^I + [\alpha_p^I, \alpha_{-p}^I]$ , which gives

$$L_0^\perp = \alpha' p^I p^I + \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I,$$

where the normal ordering constant has been put into

$$2\alpha' p^- = \frac{1}{p^+} (L_0^\perp + a), \quad a = \frac{1}{2} (D-2) \sum_{p=1}^{\infty} p.$$

This leads to

$$M^2 = \frac{1}{\alpha'} \left( a + \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I \right).$$

The divergent sum for  $a$  is regulated by using  $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$  and analytically continuing to get  $\zeta(-1) = -1/12$ . So

$$a = -\frac{1}{24}(D-2).$$

• Virasoro generators and algebra (corresponds to worldsheet energy-momentum tensor). Since  $(\alpha_n^I)^\dagger = \alpha_{-n}^I$ , get  $L_n^\perp \dagger = L_{-n}^\perp$ . Also,

$$[L_m^\perp, \alpha_n^I] = -n\alpha_{n+m}^I.$$

$$[L_m^\perp, L_n^\perp] = (m-n)L_{n+m}^\perp + \frac{D-2}{12}(m^3-m)\delta_{m+n,0}.$$

• Spacetime Lorentz symmetry corresponds to conserved currents on worldsheet, with conserved charges

$$M_{\mu\nu} = \int_0^\pi (X_\mu \mathcal{P}_\nu^\tau - (\mu \leftrightarrow \nu)) d\sigma.$$

Plug in  $\mathcal{P}_\nu^\tau = \frac{1}{2\alpha'} \dot{X}^\mu$  and plug in oscillator expansion of  $X^\mu$  to get

$$M^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu).$$

In light cone gauge, have to be careful with  $M^{-I}$ , since  $X^-$  is constrained to something complicated,

$$X^-(\tau, \sigma) = x_0^- + \sqrt{2\alpha'} \alpha_0^- \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^- e^{-in\tau} \cos n\sigma, \quad \sqrt{2\alpha'} \alpha_n^- = \frac{1}{p^+} L_n^\perp.$$

and also careful to ensure that  $[M^{-I}, M^{-J}] = 0$ . Find, after appropriately ordering terms,

$$M^{-I} = x_0^- p^I - \frac{1}{4\alpha' p^+} (x_0^I (L_0^\perp + a) + (L_0^\perp + a) x_0^I) - \frac{i}{\sqrt{2\alpha' p^+}} \sum_{n=1}^{\infty} \frac{1}{n} (L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp).$$

Then get

$$[M^{-I}, M^{-J}] = -\frac{1}{\alpha' p^{+2}} \sum_{m=1}^{\infty} (\alpha_{-m}^I \alpha_m^J - (I \leftrightarrow J)) [m(1 - ((D-2)/24)) + m^{-1}(((D-2)/24) + a)].$$

Since this must be zero, get  $D = 26$  and  $a = -1$ .

The worldsheet Hamiltonian is thus

$$H = 2\alpha' p^+ p^- = L_0^\perp - 1.$$

- The states are obtained as

$$|\lambda\rangle = \prod_{n=1}^{\infty} \prod_{I=2}^{25} (a_n^{I\dagger})^{\lambda_{n,I}} |p^+, \vec{p}_T\rangle.$$

These states are eigenstates of

$$M^2 = \frac{1}{\alpha'}(-1 + N^\perp), \quad N^\perp \equiv \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I,$$

with eigenvalues

$$M^2 = \frac{1}{\alpha'}(-1 + N^\perp), \quad N^\perp = \sum_n \sum_I n \lambda_{n,I}.$$

The groundstate is tachyonic (!). The first excited state is a massless spacetime vector with  $D - 2$  polarizations, i.e. a massless gauge field, like the photon (but in  $D = 26$ )!

The tachyon is related to the fact that the D25 brane is unstable, it decays to the closed string vacuum. The closed bosonic string is also unstable, as we'll see next time. These instabilities can be cured by adding fermions and considering the superstring. Then the critical spacetime dimension is  $D = 10$ .

The eigenstates satisfy the worldsheet SE:

$$i \frac{\partial}{\partial \tau} |\lambda\rangle = H |\lambda\rangle = (L_0^\perp - 1) |\lambda\rangle.$$

Writing  $x^+ = 2\alpha' p^+ \tau$ , this becomes

$$\left( i \frac{\partial}{\partial x^+} - \frac{1}{2p^+} (p^I p^I + M^2) \right) \phi_\lambda(x^+, p^+, \tau),$$

which is the KG (or generalization) wave equation for the corresponding field in spacetime.