5/24/18 Lecture outline

⋆ Reading: Zwiebach chapters 9 and 10.

• Continue where we left off last time: we will consider quantization of fields and then strings.

Recall from last time: classical scalar field theory, with $S = \int d^D x(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2)$. The EOM is the Klein-Gordon equation

$$(\partial^2 - m^2)\phi = 0, \qquad \partial^2 \equiv -\frac{\partial^2}{\partial t^2} + \nabla^2$$

The Hamiltonian is $H = \int d^{D-1}x(\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2)$, where $\Pi = \partial\mathcal{L}/\partial(\partial_0\phi) = \partial_0\phi$. Take e.g. $D = 1$ and get SHO with $q \to \phi$ and $m \to 1$ and $\omega \to m$.

Classical plane wave solutions: $\phi(t, \vec{x}) = ae^{-iEt+i\vec{p}\cdot\vec{x}} + c.c.$, where $E = E_p = \sqrt{\vec{p}^2 + m^2}$, and the $+c.c.$ is to make ϕ real. Letting $\phi(x) = \int \frac{d^D p}{(2\pi)^D} e^{ip\cdot x}\phi(p)$, the reality condition is $\phi(p)^* = \phi(-p)$ and the EOM is $(p^2 + m^2)\phi(p) = 0$.

• Now consider light cone gauge coordinates. Replace $\partial^2 \to -2\partial_+\partial_- + \partial_I\partial_I$ and Fourier transform

$$\phi(x^+, x^-, \vec{x}_T) = \int \frac{dp^+}{2\pi} \int \frac{d^{D-2}\vec{p}_T}{(2\pi)^{D-2}} e^{-ix^-p^+ + i\vec{x}_T\cdot\vec{p}_T}\phi(x^+, p^+, \vec{p}_T).$$

Then the EOM becomes

$$(i\frac{\partial}{\partial x^+} - \frac{1}{2p^+}(p^I p^I + m^2))\phi(x^+, p^+, \vec{p}_T) = 0.$$

Looks like the non-relativistic Schrodinger equation, with x_+ playing the role of time and p^+ playing the role of mass, even though it is secretly relativistic.

• Let's quantize! Replace ϕ with an operator. Consider

$$\phi(t, \vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \frac{1}{\sqrt{2E_p}}(a_{\vec{p}}(t)e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}(t)e^{-ip\cdot x}).$$

If we're in a spatial box, then $p_i L_i = 2\pi n_i$. Compute the energy to find

$$H = \sum_{\vec{p}>0}(\frac{1}{2E_p}\dot{a}^\dagger_p\dot{a}_p(t) + \tfrac{1}{2}E_p a^\dagger_p a_p) = \sum_{\vec{p}} E_p a^\dagger_{\vec{p}} a_{\vec{p}}.$$

where the EOM were used in the last step: $a_{\vec{p}}(t) = a_{\vec{p}}e^{-iE_pt} + a^\dagger_{-\vec{p}}e^{iE_pt}$. Also,

$$\vec{P} = \sum_{\vec{p}} \vec{p} a^\dagger_{\vec{p}} a_{\vec{p}}.$$

As expected, H and $\vec{P}$ are independent of t. We quantize this as a (complex) SHO for each value of $\vec{p}$:

$$[a_p, a_k^\dagger] = \delta_{p,k}, \qquad [a_p, a_k] = [a_p^\dagger, a_k^\dagger] = 0.$$

and interpret the above H and $\vec{P}$ has saying that $a_{\pm\vec{p}}^\dagger$ is a creation operator, creating a state with energy $E_p = \sqrt{\vec{p}^2 + m^2}$ and spatial momentum $\vec{p}$ from the vacuum $|\Omega\rangle$. (Note that we dropped the $2 \cdot \frac{1}{2} E_p$ groundstate energy contribution, for no good reason. This is the road to the unresolved cosmological constant problem, so we won't go there.)

• Now consider the Maxwell field A^μ and quantize → photons. In the vacuum, setting $j^\mu = 0$, we have $\partial_\mu F^{\nu\mu} = 0$, which implies $\partial^2 A^\mu - \partial^\mu(\partial \cdot A) = 0$. Massless. Fourier transform to $A^\mu(p)$, with $A^\mu(-p) = A^\mu(p)^*$, and get $(p^2\eta^{\mu\nu} - p^\mu p^\nu)A_\nu(p) = 0$. Gauge invariance $\delta A_\mu(p) = ip_\mu \epsilon(p)$. In light cone gauge, since $p^+ \neq 0$, can set $A^+(p) = 0$. Then get $A^- = (p^I A^I)/p^+$, i.e. A^- is not an independent d.o.f., but rather constrained, and the Maxewell EOM gives $p^2 A^\mu(p) = 0$. For $p^2 \neq 0$, require $A^\mu(p) = 0$, and for $p^2 = 0$ get that there are $D - 2$ physical transverse d.o.f., the $A^I(p)$. The one-photon states are

$$\sum_{I=2}^{D-1} \xi_I a_{p^+, p_T}^{I\dagger} |\omega\rangle.$$

• Gravitational light cone gauge conditions: $h^{++} = h^{+-} = h^{+I} = 0$. Other light cone components are constrained. So physical d.o.f. are specified by a traceless symmetric matrix h^{IJ} in the $D-2$ transverse directions. So $\frac{1}{2}D(D-3)$ d.o.f..

• Recall the relativistic point particle, with $S = \int L d\tau$ and $L = -m\sqrt{-\dot{x}^2}$, where $\dot{\equiv} \frac{d}{d\tau}$. ($\tau$ is taken to be dimensionless.) The momentum is $p_\mu = \partial L/\partial \dot{x}^\mu = m\dot{x}_\mu/\sqrt{-\dot{x}^2}$ and the EOM is $\dot{p}_\mu = 0$. In light cone gauge we take $x^+ = p^+\tau/m^2$. Then $p^+ = m\dot{x}^+/\sqrt{-\dot{x}^2}$ and the light cone gauge condition implies $\dot{x}^2 = -1/m^2$, so $p_\mu = m^2 \dot{x}_\mu$. Also, $p^2 + m^2 = 0$ yields $p^- = (p^I p^I + m^2)/2p^+$, which is solved for p^- and then $\dot{x}^- = p^-/m^2$ is integrated to $x^- = p^-\tau/m^2 + x_0^-$. Also, $x^I = x_0^I + p^I\tau/m^2$. The dynamical variables are (x^I, x_0^-, p^I, p^+).
• Heisenberg picture: put time dependence in the operators rather than the states, with $[q(t), p(t)] = i$ and

$$i\frac{d}{dt}\mathcal{O}(t) = i\frac{\partial \mathcal{O}}{\partial t} + [\mathcal{O}, H].$$

For time independent Hamiltonian, we have $|\psi(t)\rangle_S = e^{-iHt}|\psi\rangle_H$ and $\mathcal{O}_H = e^{iHt}\mathcal{O}_S e^{-iHt}$.

• Quantize the point particle in light cone gauge by taking the independent operators (x^I, x_0^-, p^I, p^+), with $[x^I, p^J] = i\eta^{IJ}$ and $[x_0^-, p^+] = i\eta^{-+} = -i$. These commutators are for either S or H picture, with the operators being functions of τ in the H picture.

The remaining variables are defined by $x^+(\tau) = p^+\tau/m^2$, $x^-(\tau) = x_0^- + p^-\tau/m^2$, $p^- = (p^I p^I + m^2)/2p^+$ (the first two are explicitly τ dependent even in the S picture).

The Hamiltonian is $\sim p^-$, which generates $\frac{\partial}{\partial x^+}$ translations. Since $\frac{\partial}{\partial \tau} = \frac{p^+}{m^2}\frac{\partial}{\partial x^+} \leftrightarrow \frac{p^+}{m^2}p^-$ the Hamiltonian is

$$H = \frac{p^+ p^-}{m^2} = \frac{1}{2m^2}(p^I p^I + m^2).$$

Verify e.g.

$$i\frac{d}{d\tau}p^\mu = [p^\mu, H] = 0, \qquad i\frac{dx^I}{d\tau} = [x^I, H] = i\frac{p^I}{m^2},$$

reproducing the correct EOM. Likewise, verify $\dot{x}_0^- = 0$ and $\dot{x}^+ = \partial_\tau x^+ = p^+/m^2$. The momentum eigenstates are labeled by $|p^+, p^I\rangle$ and these are also energy eigenstates, $H|p^+, p^I\rangle = \frac{1}{2m^2}(p^I p^I + m^2)|p^+, p^I\rangle$.

• Connect the quantized point particle with the excitations of scalar field theory via

$$|p^+, p^I\rangle \leftrightarrow a^\dagger_{p^+, p^I}|\Omega\rangle.$$

The S.E. of the quantum point particle wavefunction maps to the classical scalar field equations, e.g. in light cone gauge:

$$(i\partial_\tau - \frac{1}{2m^2}(p^I p^I + m^2))\phi(\tau, p^+, p^I) = 0$$

is either the quantum S.E. of the point particle or the classical field equations of a scalar field.

(Aside: the light cone is here used as a trick to get to "second quantization." "First quantization" is what you learn the first time you study (non-relativisitic) QM: replace coordinates and momenta with operators, and Poisson brackets with commutators. Second quantization is for field theory, replacing the fields and their conjugate momenta with operators, and their PBs with commutators, leading to multi-particle states. Here light-cone first quantization of the point particle leads to a Schrodinger equation that agrees with the classical EOM of a light-cone field theory, which we then need to quantize again to get second quantization.

• Lorentz transformations correspond to the inf. transformations $\delta x^\mu = \epsilon^{\mu\nu}x_\nu$, and the corresponding conserved Noether charges are the generalized angular momenta $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$. These generate rotations and boosts. Have e.g. $[M^{\mu\nu}, x^\rho] = i\eta^{\mu\rho}x^\nu - i\eta^{\nu\rho}x^\mu$ and $[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho}M^{\nu\sigma}\pm$(perms).

In light cone coordinates, have M^{IJ}, $M^{\pm I}$, M^{+-}, with e.g. $[M^{+-}, M^{+I}] = iM^{+I}$ and $[M^{-I}, M^{-J}] = 0$. There are some ordering issues for some of these, where operators which don't commute need to be replaced with symmetrized averages, e.g. $M^{+-} = -\frac{1}{2}(x_0^- p^+ + p^+ x_0^-)$ and $M^{-I} = x_0^- p^I - \frac{1}{2}(x_0^I p^- + p^- x_0^I)$.

• Open string. Imposed constraints $(\dot{X} \pm X')^2 = 0$ to get

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X^{\mu'}, \qquad \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu.$$

In light cone gauge, much as with the point particle, the independent variables are $(X^I, (\sigma) x_0^-, \mathcal{P}^{\tau I}(\sigma), p^+)$. In the H picture the capitalized ones depend (implicitly) on τ too. The commutation relations are

$$[X^I(\sigma), \mathcal{P}^{\tau J}(\sigma')] = i\eta^{IJ}\delta(\sigma - \sigma'), \qquad [x_0^-, p^+] = -i.$$

The Hamiltonian is taken to be

$$H = 2\alpha' p^+ p^- = 2\alpha' p^+ \int_0^\pi d\sigma \mathcal{P}^{\tau -} = \pi\alpha' \int_0^\pi d\sigma (\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + X^{I'} X^{I'} (2\pi\alpha')^{-2})$$

Can write $H = L_0^\perp$ since $L_0^\perp = 2\alpha' p^+ p^-$.

This H properly yields the expected time derivatives, e.g. $\dot{X}^I = 2\pi\alpha' \mathcal{P}^{\tau I}$.

Recall the solution with N BCs:

$$X^I(\tau, \sigma) = x_0^I + \sqrt{2\alpha'}\alpha_0^I \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^I \cos n\sigma e^{-in\tau}. \tag{1}$$

The needed commutators are ensured by

$$[\alpha_m^I, \alpha_n^J] = m\eta^{IJ}\delta_{n+m,0}.$$

Also, as before, we define $\alpha_0^I \equiv \sqrt{2\alpha'} p^I$. Now define $\alpha_{n>0}^\mu = \sqrt{n} a_n^\mu$ and $\alpha_{-n}^\mu = a_n^{\mu *}\sqrt{n}$ to rewrite the above as

$$[a_m^I, a^{J^\dagger}] = \delta_{m,n}\eta^{IJ}. \tag{2}$$

• The transverse light cone coordinates can be described by

$$S_{l.c.} = \int d\tau d\sigma \frac{1}{4\pi\alpha'} (\dot{X}^I \dot{X}^I - X^{I'} X^{I'}).$$

Gives correct $\mathcal{P}^{\tau I} = \partial\mathcal{L}/\partial\dot{X}^I$ and correct $H = \int d\sigma (\mathcal{P}^{\tau I}\dot{X}^I - \mathcal{L})$.

Writing $X^I(\tau,\sigma) = q^I(\tau) + 2\sqrt{\alpha'}\sum_{n=1}^{\infty} q_N^I(\tau) n^{-1/2}\cos n\sigma$ and plugging into the action above gives

$$S = \int d\tau [\frac{1}{4\alpha'}\dot{q}^I\dot{q}^I + \sum_{n=1}^{\infty}(\frac{1}{2n}\dot{q}_n^I\dot{q}_n^I - \frac{n}{2}q_n^I q_n^I)]$$

and

$$H = \alpha' p^I p^I + \sum_{n=1}^{\infty}\frac{n}{2}(p_n^I p_n^I + q_n^I q_m^I).$$

A bunch of harmonic oscillators. Relate to (1) and (2), showing that the a_m can be interpreted as the usual harmonic oscillator annihilation operators.

- $X^+(\tau\sigma) = 2\alpha' p^+\tau = \sqrt{2\alpha'}\alpha_0^+\tau$. For X^- recall expansion, with $\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n^\perp$, where $L_n^\perp \equiv \frac{1}{2}\sum_p \alpha_{n-p}^I\alpha_p^I$ is the transverse Virasoro operator. There is an ordering ambiguity here, only for $L_0^\perp$:

$$L_0^\perp = \tfrac{1}{2}\alpha_0\alpha_0 + \tfrac{1}{2}\sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I + \tfrac{1}{2}\sum_{p=1}^{\infty}\alpha_p^I\alpha_{-p}^I.$$