5/22/18 Lecture outline

⋆ Reading: Zwiebach chapters 9 and 10.

• Continue where we left off last time: we saw that we can generalize static gauge to

$$n \cdot \mathcal{P}^\sigma = 0, \qquad n \cdot X = \beta\alpha'(n \cdot p)\tau, \qquad n \cdot p = \frac{2\pi}{\beta} n \cdot \mathcal{P}^\tau,$$

where $\beta = 2$ for open strings and $\beta = 1$ for closed strings. These lead to

$$\dot{X} \cdot X' = 0 \qquad \dot{X}^2 + c^2 X'^2 = 0. \tag{1}$$

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu \qquad \mathcal{P}^{\sigma\mu} = -\frac{c^2}{2\pi\alpha'} X^{\mu'}, \tag{2}$$

$$(\partial_\tau^2 - c^2\partial_\sigma^2)X^\mu = 0. \tag{3}$$

The general solution of the linear equations (3) is a superposition of Fourier modes

$$X^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}^{\infty} \frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma,$$

where $\alpha_{-n}^\mu \equiv \alpha_n^{\mu *}$ (to make X^μ real) and it's also convenient to define $\alpha_0^\mu \equiv \sqrt{2\alpha'}p^\mu$. Then

$$\dot{X}^\mu \pm X^{\mu'} = \sqrt{2\alpha'}\sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in(\tau\pm\sigma)}.$$

• In light cone gauge take $n_\mu = (1/\sqrt{2}, 1/\sqrt{2}, 0, \ldots)$. Then $n \cdot X = X^+$ and $n \cdot p = p^+$, so our constraint gives $X^+ = \beta\alpha' p^+ \tau$ and $p^+ = 2\pi\mathcal{P}^{\tau +}/\beta$ (again, $\beta = 2$ for open strings and $\beta = 1$ for closed strings. Also note, $X'^+ = 0$ and $\dot{X}^+ = \beta\alpha' p^+$); of course, p^+ is a constant of the motion. Since the constraints give $(\dot{X} \pm X')^2 = -2(\dot{X}^+ \pm X'^+)(\dot{X}^- \pm X'^-) + (\dot{X}^I \pm X'^I)^2 = 0$, we can write this as $\partial_\tau X^- \pm \partial_\sigma X^- = \frac{1}{\beta\alpha'}\frac{1}{2p^+}(\dot{X}^I \pm X^{I'})^2$, where I are the transverse directions. This leads to

$$\sqrt{2\alpha'}\alpha_n^- \equiv \frac{1}{p^+}L_n^\perp, \qquad L_n^\perp = \tfrac{1}{2}\sum_{m=-\infty}^{\infty} \alpha_{n-m}^I \alpha_m^I.$$

Note that the worldsheet coordinates dropped out. This means that there is no dynamics in X^-, other than the zero mode. For $n = 0$, using $\alpha_0^- = \sqrt{2\alpha'}p^-$ get $2\alpha' p^+ p^- = L_0^\perp$. Light cone gauge allows us to make $\dot{X}^+$ a constant, and to solve for the derivatives of X^- (without having to take a square root). Finally, note that the string has

$$M^2 = -p^2 = 2p^+p^- - P^I p^I = \frac{1}{\alpha'}\sum_{n=1}^{\infty}\alpha_n^{I*}\alpha_n^I.$$

See that all classical states have $M^2 \geq 0$. Note that this is a continuous $M \geq 0$ classically, and only one state with $M = 0$; neither would be good for using strings as elementary particles. As we will soon discuss, quantum effects cures both of these problems: the values of M will be quantized, and the states with $M = 0$ will give the expected polarizations for a massless gauge field in the open string case, and for a gravity (plus extra stuff) in the closed string case.

- We will consider quantization of fields and then strings. As a warmup, consider classical scalar field theory, with $S = \int d^D x(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2)$. The EOM is the Klein-Gordon equation

$$(\partial^2 - m^2)\phi = 0, \qquad \partial^2 \equiv -\frac{\partial^2}{\partial t^2} + \nabla^2$$

The Hamiltonian is $H = \int d^{D-1}x(\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2)$, where $\Pi = \partial\mathcal{L}/\partial(\partial_0\phi) = \partial_0\phi$. Take e.g. $D = 1$ and get SHO with $q \to \phi$ and $m \to 1$ and $\omega \to m$.

Classical plane wave solutions: $\phi(t, \vec{x}) = ae^{-iEt+i\vec{p}\cdot\vec{x}} + c.c.$, where $E = E_p = \sqrt{\vec{p}^2 + m^2}$, and the $+c.c.$ is to make ϕ real. Letting $\phi(x) = \int \frac{d^D p}{(2\pi)^D} e^{ip\cdot x}\phi(p)$, the reality condition is $\phi(p)^* = \phi(-p)$ and the EOM is $(p^2 + m^2)\phi(p) = 0$.