5/10/18 Lecture outline

⋆ Reading: Zwiebach chapter 8

• Symmetries and conservation laws. Recall charge conservation $\partial_\mu j^\mu = 0$, which is required by gauge invariance of $\mathcal{L} \supset A_\mu j^\mu$, i.e. $\delta\mathcal{L} = 0$ under $\delta A_\mu = \partial_\mu f$. Show that it implies conservation of $Q = \int d^3x j^0$.

• Recall Noether's theorem for $L(q, \dot{q})$: if continuous symmetry δq_i then $p_i \delta q_i$ is conserved.

• Likewise, for $S = \int d\xi^0 \dots d\xi^k \mathcal{L}(\phi^a, \partial_\alpha \phi^a)$, a symmetry $\delta\phi^a$ implies a conserved current $j^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \delta\phi^a$: show it satisfies $\partial_\alpha j^\alpha = 0$, so $Q = \int d\xi^1 \dots d\xi^k$ has $\frac{d}{d\xi^0} Q = 0$.

For a string $S = \int d\xi^0 d\xi^1 \mathcal{L}(\partial_\alpha X^\mu)$ (has translation invariance, $\delta X^\mu = \epsilon^\mu$ so there is a conserved current $\epsilon^\mu j^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \delta X^\mu$. So get conservation of $j^a_\mu = \mathcal{P}^a_\mu$ (where $a = \sigma, \tau$) is the conserved Noether current for spacetime translation invariance, $\delta X^\mu = \epsilon^\mu$. The string equations of motion are equivalent to the worldsheet conservation of this current: $\partial_a j^a_\mu = 0$. The spacetime momentum of the string is the corresponding conserved charge: $p_\mu = \int d\sigma \mathcal{P}^\tau_\mu$. So $\frac{dp_\mu}{d\tau} = -\int_0^{\sigma_1} \partial_\sigma \mathcal{P}^\sigma_\mu = -\mathcal{P}^\sigma_\mu|_0^{\sigma^1}$. It is conserved for the closed string, or open Neumann BCs. Not conserved for Dirichlet BCs. The Dirichlet case means that the string ends on a D-brane, and momentum can go through the string into the D-brane (their total momentum is conserved). Same for wave on a string with the ends tied down, e.g. a traveling wave is reflected, which flips $p \to -p$, but the difference in momentum is transferred to the post at the end and total momentum of the system is conserved.

• Note that p_μ is a conserved *worldsheet* charge. It becomes a conserved spacetime charge in static gauge, $\tau = t$. We can write more generally the conserved flux of worldsheet current as $(\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu) \cdot (d\sigma, -d\tau)$, where $(d\tau, d\sigma)$ is the tangent to the curve Γ that we're integrating over and $(d\sigma, -d\tau)$ gives the outward normal. So $p_\mu(\Gamma) = \int_\Gamma (\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau)$. The difference between some Γ and Γ' with the same endpoints (i.e. $\partial(\Gamma - \Gamma') = 0$) is $\oint_{\Gamma - \Gamma' = \partial R} (\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau) = \int_R d\tau d\sigma (\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu) = 0$.

• Using the $\mathcal{P}^{\alpha\mu}$ that we found last week in static gauge we get for the conserved charges

$$p^0 = \frac{E}{c} = \int \frac{T_0 ds}{\sqrt{1 - v_\perp^2/c^2}}, \qquad \vec{p} = \int \frac{T_0 ds}{c^2} \frac{v_\perp}{\sqrt{1 - v_\perp^2/c^2}}.$$

• Lorentz symmetry comes from the worldsheet symmetry $\delta X^\mu = \epsilon^{\mu\nu} X_\nu$, which is a symmetry if $\epsilon^{\mu\nu} = \epsilon^{[\mu\nu]}$, e.g. $\delta(\eta_{\mu\nu} X^\mu X^\nu) = 0$. The terms in the string Lagrangian $\mathcal{L}_{NG} = -\frac{T_0}{c}\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}$ involve $\eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu$, which again is invariant under $\delta X^\mu = \epsilon^{\mu\nu} X_\nu$.

The associated conserved currents are $\mathcal{M}^{\alpha}_{\mu\nu} = X_{\mu}\mathcal{P}^{\alpha}_{\nu} - (\mu \leftrightarrow \nu)$. The corresponding charges $M_{\mu\nu} = \int(\mathcal{M}^{\tau}_{\mu\nu} d\sigma - \mathcal{M}^{\sigma}_{\mu\nu} d\tau)$ are the angular momenta. Note that $M^{0i} = ctp^i - \int d\sigma X^i \mathcal{P}^{\tau 0}$, which can be interpreted as $X^i_{cm}(t) = \frac{-cM^{0i}}{E} + t\frac{c^2 p^i}{E}$.