5/3/18 Lecture outline

⋆ Reading: Zwiebach chapters 7

• Recall from last time. We chose $\tau = t$ (static gauge) and obtained

$$L_{NG} = -T_0 \int ds \sqrt{1 - v_\perp^2/c^2},$$

where $ds \equiv |d\vec{X}|_{t=const} = |\partial_\sigma \vec{X}||d\sigma|$ is the length element of the string and $\vec{v}_\perp = \partial_t \vec{X} - (\partial_t \vec{X} \cdot \partial_s \vec{X})\partial_s \vec{X}$. This gives

$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c^2} \frac{(\partial_s \vec{X} \cdot \partial_t \vec{X})\dot{X}^\mu + (c^2 - (\partial_t \vec{X})^2)\partial_s X^\mu}{\sqrt{1 - v_\perp^2/c^2}},$$

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2} \frac{ds}{d\sigma} \frac{\dot{X}^\mu - (\partial_s \vec{X} \cdot \partial_t \vec{X})\partial_s X^\mu}{\sqrt{1 - v_\perp^2/c^2}}.$$

which for the open string with Neumann BCs implies that the ends move at $v = v_\perp = c$.

Our goal is to motivate that we can chose $(\tau\sigma)$ such that

$$\dot{X} \cdot X' = 0 \qquad \dot{X}^2 + X'^2 = 0. \tag{1}$$

The first steps were discussed in the previous lecture.

• Step 2 (Z, chapter 7): we can choose σ such that $\partial_\sigma \vec{X} \cdot \partial_t \vec{X} = 0$ along entire string (we saw it above for Neumann endpoints). The interpretation is that we take the timelike and spacelike vectors $\dot{X}^\mu$ and X'^μ to be orthogonal. This gives $\vec{v}_\perp = \vec{v} \equiv \dot{\vec{X}}$ along the entire string. Then $\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\frac{ds}{d\sigma}\gamma \partial_t X^\mu$ and $\mathcal{P}^{\sigma\mu} = -T_0 \gamma^{-1} \partial_s X^\mu$, with $\gamma \equiv 1/\sqrt{1 - v_\perp^2/c^2}$.

Now consider the $\mu = 0$ component of the EOM: $\partial_t \mathcal{P}^{\tau\mu} = -\partial_\sigma \mathcal{P}^{\sigma\mu}$, which for $\mu = 0$ gives that $(T_0/c)\frac{ds}{d\sigma}\gamma$ is a constant of the motion. Indeed this is proportional to the energy of an element of string. In a HW you will show that the string Hamiltonian is indeed $H = \int T_0 ds/\sqrt{1 - v_\perp^2/c^2}$.

Now the space components of the EOM can be written as $\mu_{eff}\partial_t \vec{v}_\perp = \partial_s(T_{eff}\partial_s \vec{X})$, with $T_{eff} = T_0/\gamma$ and $\mu_{eff} = T_0\gamma/c^2$.

• Since $\frac{ds}{d\sigma}\gamma$ is a constant, we can choose our σ parameterization to set it equal to 1. So $(\frac{ds}{d\sigma})^2 + c^{-2}v_\perp^2 = 1$. This can be written as the constraint: $(\partial_\sigma \vec{X})^2 + (\partial_{X_0} \vec{X})^2 = 1$.

• Summary: choose σ parameterization such that

$$\partial_\sigma \vec{X} \cdot \partial_\tau \vec{X} = 0 \qquad \text{and} \qquad d\sigma = \frac{ds}{\sqrt{1 - v_\perp^2/c^2}} = \frac{dE}{T_0}.$$

(Using $H = \int T_0 ds/\sqrt{1 - v_\perp^2/c^2}$ and $\partial_t(ds/\sqrt{1-v_\perp^2/c^2}) = 0$.) The last equation above is equivalent to $(\partial_\sigma \vec{X})^2 + c^{-2}(\partial_t \vec{X})^2 = 1$. With this worldsheet gauge choice,

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\partial_t X^\mu = \frac{T^0}{c^2}(c, \vec{v}_\perp), \qquad \mathcal{P}^{\sigma,\mu} = -T_0 \partial_\sigma X^\mu = (0, -T_0 \partial_\sigma \vec{X}).$$

We can write this as

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu \qquad \mathcal{P}^{\sigma\mu} = -\frac{c^2}{2\pi\alpha'}X^{\mu'}, \tag{2}$$

and then the EOM is simply a linear wave equation, and we also need to impose the constraints:

$$(\partial_\tau^2 - c^2\partial_\sigma^2)X^\mu = 0, \qquad (\dot{X} \pm X')^2 = 0. \tag{3}$$

• Solution of the EOM for open string with free (N) BCs at each end. Write the solution of the EOM as $\vec{X}(t,\sigma) = \frac{1}{2}(\vec{F}(ct+\sigma) + G(ct-\sigma))$. The BC at $\sigma = 0$ gives $F'(ct) = G'(ct)$, which implies $G = F + const$, and the constant can be absorbed into F so $\vec{X}(t,\sigma) = \frac{1}{2}(\vec{F}(ct+\sigma) + \vec{F}(ct-\sigma))$ where the open string has $\sigma \in [0, \sigma_1]$ and (1) implies that $|\frac{d\vec{F}(u)}{du}|^2 = 1$, and $\vec{X}'|_{ends} = 0$ implies $\vec{F}(u + 2\sigma_1) = \vec{F}(u) + 2\sigma_1 \vec{v}_0/c$. Note $\vec{F}(u)$ is the position of the $\sigma = 0$ end at time u/c. Then show that $\vec{v}_0$ is the average velocity of any point σ on the string over time interval $2\sigma_1/c$. Observing motion of $\sigma = 0$ end over that Δt, together with E, gives motion of string for all t. Example from book: $\vec{X}(t, \sigma = 0) = \frac{\ell}{2}(\cos\omega t, \sin\omega t)$. Find $\vec{F}(u) = \frac{\sigma_1}{\pi}(\cos \pi u/\sigma_1, \sin \pi u/\sigma_1)$, with $\vec{v}_0 = 0$. $|\frac{d\vec{F}}{du}|^2 = 1$ gives $\ell = 2c/\omega = 2E/\pi T_0$. Finally, $\vec{X}(t,\sigma) = \frac{\sigma_1}{\pi}\cos(\pi\sigma/\sigma_1)(\cos(\pi ct/\sigma_1), \sin(\pi ct/\sigma_1))$.

• Closed string motion: again, solve the string worldsheet wave equation by $\vec{X} = \frac{1}{2}(\vec{F}(u) + \vec{G}(v))$ where $u = ct + \sigma$ and $v = ct - \sigma$. The parameterization constraints give $|\vec{F}'(u)|^2 = |\vec{G}'(v)|^2 = 1$ and $\sigma \sim \sigma + \sigma_1$ periodicity, with $\sigma_1 = E/T_0$.