5/1/13 Lecture outline

⋆ Reading: Zwiebach chapter 6.

• Recall, $S_{string} = \int dtdx\mathcal{L}(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x})$, with momentum densities

$$\mathcal{P}^t = \frac{\partial \mathcal{L}}{\partial \dot{y}}, \qquad \mathcal{P}^x = \frac{\partial \mathcal{L}}{\partial y'}.$$

Least action gives the equations of motion

$$\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0.$$

$S_{rel,part} = -mc\int ds + \frac{q}{c}\int A_\mu dx^\mu$, where the first term is the worldline length. For a string, we use the worldsheet area.

• For a string world-sheet, we need two parameters, ξ^a, $a = 1, 2$. The string trajectory is $x : \Sigma \to M$, where Σ is the 2d world-sheet, with local coordinates ξ^a, and M is the target space, with local coordinates x^μ. The worldsheet area element is $A = \int d^2\xi\sqrt{|h|}$, where h_{ab} is the worldsheet metric, and $|h|$ is its determinant. Suppose that the target space has metric $g_{\mu\nu}$, with space-time length e.g. $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. By writing $dx^\mu = \partial_a x^\mu d\xi^a$, we get

$$ds^2 = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b}d\xi^a d\xi^b, \qquad \text{so} \qquad h_{ab} = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b},$$

where this h_{ab} is called the induced metric. So the worldsheet area functional is

$$A = \int d^2\xi\sqrt{\det(g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b})}.$$

• For strings in Minkowski spacetime, we write it instead as $X^\mu(\tau, \sigma)$. There is also a needed minus sign, as the area element is $\sqrt{|g|}$, actually involves the absolute value of the determinant, and the determinant is negative (just like $\det\eta = -1$). So

$$A = \int d\tau d\sigma\sqrt{(\frac{\partial X}{\partial \tau}\cdot\frac{\partial X}{\partial \sigma})^2 - (\frac{\partial X}{\partial \tau})^2(\frac{\partial X}{\partial \sigma})^2},$$

where the spacetime indices are contracted with the metric $g_{\mu\nu}$. To get an action with $[S] = ML^2/T$, we have

$$S_{Nambu-Goto} = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau\int d\sigma\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2},$$

where we define $\dot{X}^\mu \equiv \frac{dx^\mu}{d\tau}$ and $X^{\mu\prime} \equiv \frac{\partial X^\mu}{\partial \sigma}$ annd T_0 is the string tension, with $[T_0] = [F] = [ML/T^2]$.

The action is reparameterization invariant: can take $(\tau, \sigma) \to (\tau'(\tau, \sigma), \sigma'(\tau, \sigma))$ and get $S \to S$. Enormous symmetry/redundancy in choice of (τ, σ); can "fix the gauge" to some convenient choice, and the physics is completely independent of the choice.

• We can write S_{NG} in terms of the Lagrangian density

$$\mathcal{L}_{NG} = -\frac{T_0}{c}\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2},$$

and we have

$$\mathcal{P}^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}},$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}.$$

The condition $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0.$$

For the open string, $\delta S = 0$ also requires $\int d\tau [\delta X^\mu P^\sigma_\mu]_0^{\sigma_0} = 0$, which requires for each μ index either of the Dirichlet or Neumann BCs, at each end:

$$\text{Dirichlet} \qquad \frac{\partial X^\mu}{\partial \tau}(\tau, \sigma_*) = 0 \qquad \to \qquad \delta X^\mu(\tau, \sigma_*) = 0,$$

$$\text{Neumann} \qquad \mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0.$$